The Extended Predicative Mahlo Universe in Explicit Mathematics

Anton Setzer

(Joint work with Reinhard Kahle)

Swansea University, Swansea UK

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Proof Theoretic Programme

- Hilbert's program.
  - Proof consistency of mathematical theories by finitary methods.
- Doesn't work because of Gödel's Incompleteness theorem.
- Gentzen: Reduction of consistency to well-foundedness of ordinal notation systems.
- For weaker theories gives some insight.
- Direct insight from impredicative ordinal notation systems limited.

Proof Theoretic Programme

- Instead: replace in Hilbert's program "finitary method" by
  - "reduction to a theory with some insight into its consistency".
  - Or by
    "reduction to a theory which formulates the reason why we believe in its consistency".
  - Different approaches possible.
  - Most successful approach: constructive theories.
  - Candidates could be
    - Frege structures,
    - Feferman's systems of explicit mathematics
    - Martin-Löf Type Theory.
  - Most effort has been taken to develop Martin-Löf Type Theory for that purpose.
  - Here we will see that Feferman's systems of explicit mathematics has some advantages.

Development of Advanced Data Structures

- Needed: development of predicatively justified strong extensions of Martin-Löf Type Theory.
- Benefits outside this programme:
  - Discovery of advanced data structures for use in programming.

Martin-Löf Type Theory

- In Martin-Löf Type Theory we have:
  - Terms and types, with notation

$$a : A \qquad A : \mathrm{Type}$$

    for a of type A and A is a type
  - Dependent function types, with notation

$$(x : A) \to B(x)$$

    for the type of functions mapping $a : A$ to an element of $B(a)$,
  - $$\mathrm{Set} : \mathrm{Type}$$

    where Set is the collection of small types.
    Partial functions require an ecoding.
  - Inductive data types, such as

$$\mathbb{N} : \mathrm{Set} \qquad \mathrm{List} : \mathrm{Set} \to \mathrm{Set}$$

Martin-Löf Type Theory

- Everything is total

$$f : A \to B$$

  is the set of total functions mapping any $a : A$ to B.
  Treatment of partial functions requires some encoding.
- Everything is typed, terms t can only be introduced in the form

$$t : A \text{ or } t : \mathrm{Type}$$

Explicit Mathematics – Terms

- Basic first order elements are terms.
- Terms can be applied to each other freely, so

$$\mathsf{suc}\ \mathsf{nat}$$

  is a term.
- Explicit mathematics is presented by using second order logic.
  - In order to obtain a short formulation.
- We will present it here in a style which is closer to type theory and make some minor modifications (work in progress).

Explicit Mathematics – Terms, Predictates

- We have predicates
  - $s = t, \quad s\downarrow$

    and define $s \simeq t$ as usual
  - $\Re(a)$

    for a is a name for a type
    - We define

$$\begin{array}{lcl} a \mathrel{\dot\subseteq} \Re & := & \forall x \mathrel{\dot\in} a.\Re(x) \ , \\ \Re_{\Re}(a) & := & \Re(a) \wedge a \mathrel{\dot\subseteq} \Re \ . \end{array}$$

  - $x \mathrel{\dot\in} a$

    for x is an element of the type denoted by a.
    - We have an axiom

$$x \mathrel{\dot\in} a \rightarrow \Re(a)$$

  - $f : b \mathrel{\dot\rightarrow} c := \forall x \mathrel{\dot\in} a.f\ x \mathrel{\dot\in} b.$

Explicit Mathematics – Predicates

- Possibly other predicates such as

  E.g. one can introduce N for the collection of natural numbers, where one writes

  $x \in \mathrm{N}$ for $\mathrm{N}(x)$.

Explicit Mathematics – Constants

- Constants are for instance
  - k, s

    for the k and s combinator, which allows to define untyped λ-terms (when working in explicit mathematics one uses usually just λ-terms).
  - $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$

    for pairing, first projection, second projection.
  - $\mathsf{zero}, \mathsf{suc}$

    for the elements 0 and the successor operation for natural nubmers,
  - nat

    for the term denoting the set of natural numbers and

Explicit Mathematics – Constants

- j

  where j (a, f) denotes

$$\{\mathsf{p}\ x\ y \mid x \mathrel{\dot{\in}} a \wedge y \mathrel{\dot{\in}} f\ x\}$$

Explicit Mathematics – Equality/Definedness Axioms

- We have axioms expressing definedness and partial equalities

$$\mathsf{s}\ a\ b \downarrow$$
$$\mathsf{s}\ a\ b\ c \simeq a\ c\ (b\ c)$$

$$\mathsf{p}_0\ (\mathsf{p}\ a\ b) = a$$
$$\mathsf{p}_1\ (\mathsf{p}\ a\ b) = b$$

- We write (a, b) for $\mathsf{p}\ a\ b$,
  (a, b, c) for $((a, b), c)$ etc.

Explicit Mathematics – Names and their Elements

- We have aioms defining names and elements of the sets denoted by them, e.g.
  - Natural numbers:

$$
\begin{array}{l}
\Re(\mathrm{nat}) \\
0 \mathrel{\dot{\in}} \mathrm{nat} \\
\mathrm{suc} : \mathrm{nat} \mathrel{\dot{\to}} \mathrm{nat} \\
\varphi(0) \land (\forall x.\varphi(x) \to \varphi(\mathrm{suc}\ x)) \to \forall x \mathrel{\dot{\in}} \mathrm{nat}.\phi(x)
\end{array}
$$

  - Complement:

$$
\begin{array}{l}
\Re(a) \leftrightarrow \Re(\mathrm{co}\ a) \\
\Re(a) \to (x \mathrel{\dot{\in}} \mathrm{co}\ a \leftrightarrow \neg(x \mathrel{\dot{\in}} a))
\end{array}
$$

  - Join:

$$
\begin{array}{l}
\Re(\mathrm{j}\ x) \leftrightarrow \exists a, f.x = (a, f) \land \Re(a) \land \forall y \mathrel{\dot{\in}} a.\Re(f\ \ y) \\
\Re(\mathrm{j}\ (a, f)) \to \\
\quad (x \mathrel{\dot{\in}} \mathrm{j}\ (a, f) \leftrightarrow (\exists y, z\ .\ x = (y, z) \land y \mathrel{\dot{\in}} a \land z \mathrel{\dot{\in}} f\ \ y))
\end{array}
$$

Universes in Type Theory

- Universes = collection of sets.
- Formulated in Type Theory as

$$\mathrm{U} : \mathrm{Set} \qquad \mathrm{T} : \mathrm{U} \to \mathrm{Set}$$

- U = set of codes for sets.
- T = decoding function.

Proof-Theoretically Strong Example

- U closed under $\mathbb{N}$:
  - $\widehat{\mathbb{N}} : \mathrm{U}$
  - $\mathrm{T}\ \widehat{\mathbb{N}} = \mathbb{N}$.
- U closed under dependent product type Π:
  - $\widehat{\Pi} : (a : \mathrm{U}) \to (\mathrm{T}\ a \to \mathrm{U}) \to \mathrm{U}$
  - $\mathrm{T}\ (\widehat{\Pi}\ a\ b) = (x : \mathrm{T}\ a) \to \mathrm{T}\ (b\ x)$
- Other standard universe construction.
- U closed under dependent product type W:
  - $\widehat{\mathrm{W}} : (a : \mathrm{U}) \to (\mathrm{T}\ a \to \mathrm{U}) \to \mathrm{U}$
  - $\mathrm{T}\ (\widehat{\mathrm{W}}\ a\ b) = \mathrm{W} x : \mathrm{T}\ a.\mathrm{T}\ (b\ x)$.

Strength: One recursively inaccessible $+\ \omega$ admissibles above.

Universes in Explicit Mathematics

- We have a predicate $\mathcal{U}(t)$ for t is a name for a universe.
- We axioms
  - $\mathcal{U}(u) \to \Re_{\Re}(u)$.
  - $\mathcal{U}(u)$ implies u is closed under standard universe constructions such as

$$
\begin{aligned}
&\mathsf{nat} \,\dot{\in}\, u \\
&(a \,\dot{\in}\, u \wedge (f : a \dot{\to} u)) \to \mathsf{j}\,(a, f) \,\dot{\in}\, u
\end{aligned}
$$

Steps Towards Mahlo

- First step beyond standard universe
  - The super universe (Palmgren).
  - He introduced a universe V,
  - together with a universe operator $\mathrm{U} : \mathrm{Fam}(\mathrm{V}) \to \mathrm{V}$,
    - $\mathrm{Fam}(\mathrm{V})$ is the set of families of sets in V indexed over elements of V, roughly speaking

$$\{(B_x)_{x:B} | B : \mathrm{V}, \quad x : B \Rightarrow B_x : \mathrm{V}\}$$

  - s.t. for any family of sets A in V, $\mathrm{U}(A)$ is a universe containing all elements of A.

Steps Towards Mahlo

- A Universe is a family of sets closed under constructions for forming sets.
- We can now form a universe, closed under the formation of the next universe above a family of sets.
- (The next slide doesn't exhaust the strength, it shows only universes containing one set, not universes containing family of sets)

Illustration of the Super Universe

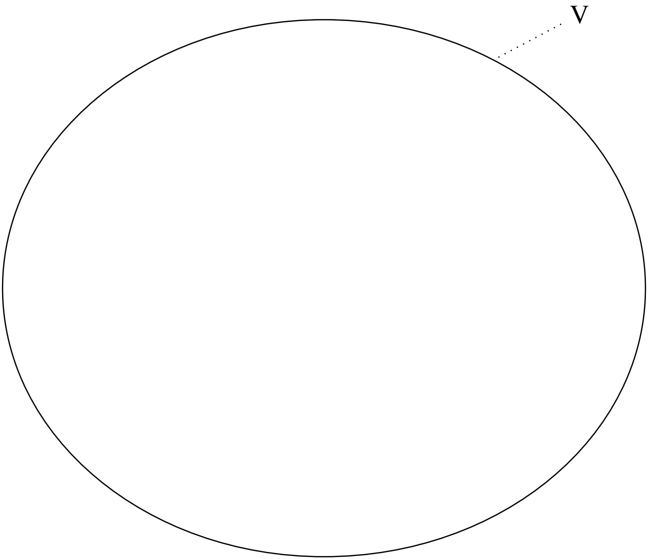
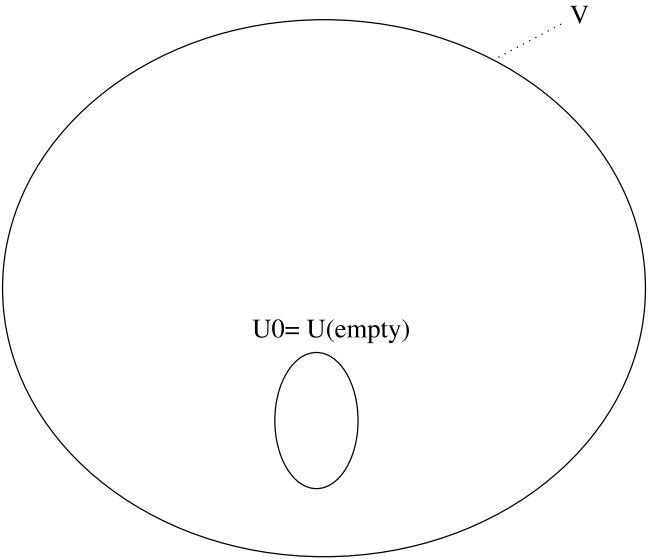


Illustration of the Super Universe



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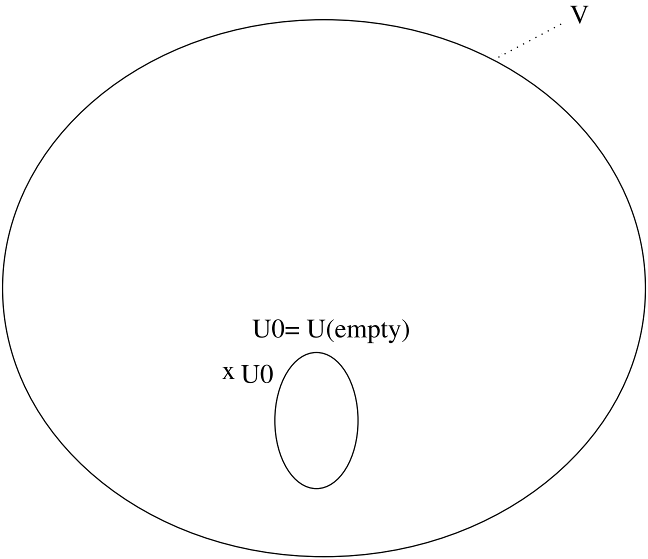

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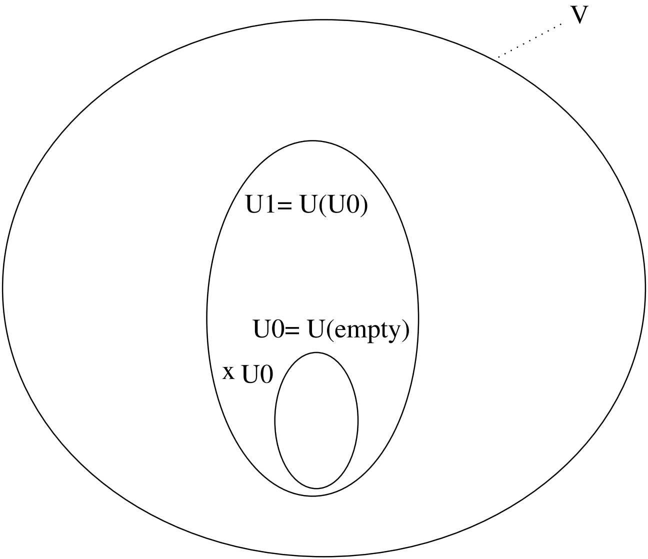

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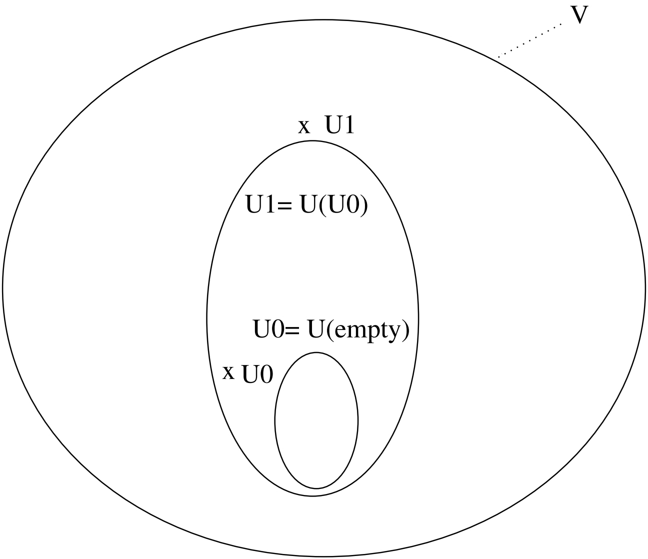

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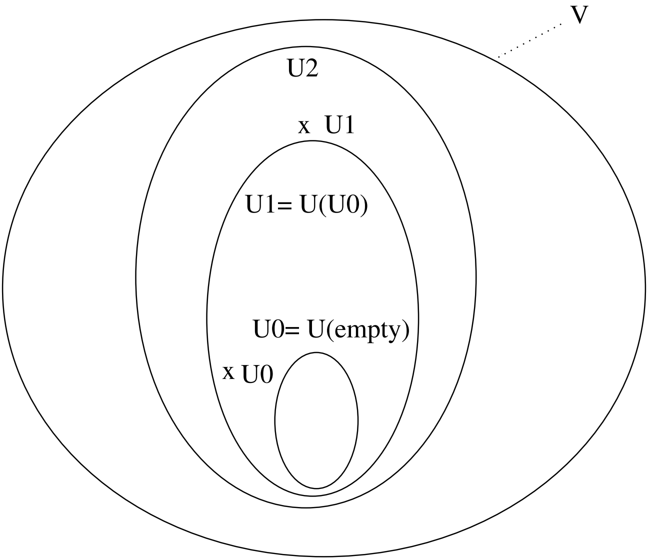

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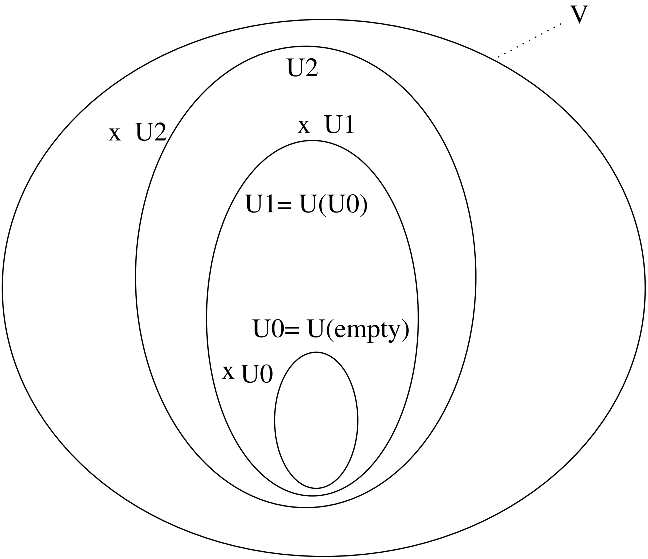


Super^n-Universes

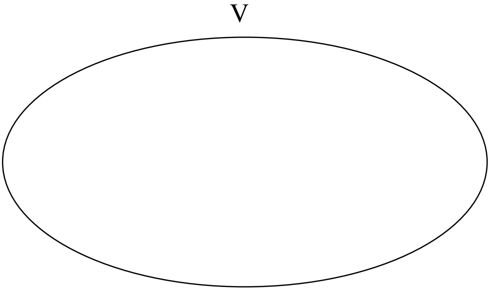
- The above can be continued: We can form a
  - super^2-universe V,
  - closed under a super-universe operator, forming a super universe above a family of sets in V.
- And we can iterate the above n-many times, and even go beyond.
- Up to now everything is inductive-recursive (a strong extension of inductive data types).

Mahlo Universe

- The Mahlo universe is
  - a universe V,
  - which has not only subuniverses corresponding to some operators, but subuniverses corresponding to all operators it is closed under:

    For every universe operator on V,
    i.e. for every $f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$,
    there exists a universe U_f closed under f.

Illustration of the Mahlo Universe



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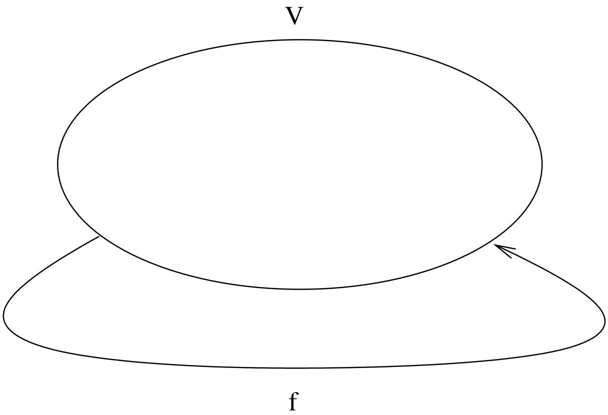

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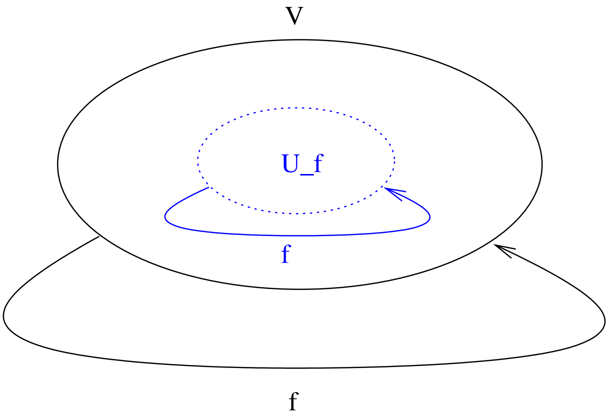

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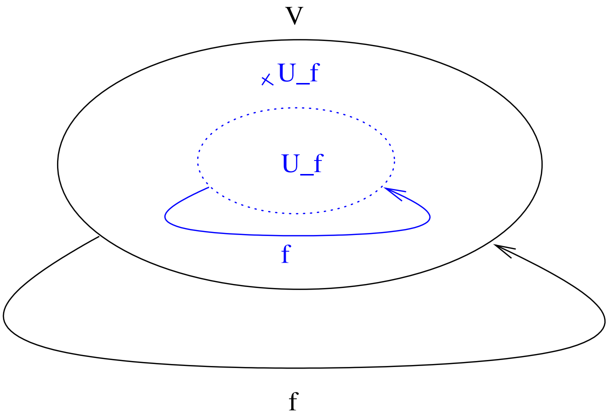


Formulation of Mahlo Universe

$$\mathrm{V} : \mathrm{Set} \qquad \mathrm{T_V} : \mathrm{V} \to \mathrm{Set}$$

Assume $(f, g) : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$ i.e.

$f : (x : \mathrm{V}) \to (\mathrm{T_V}\ x \to \mathrm{V}) \to \mathrm{V}$

$g : (x : \mathrm{V}) \to (\mathrm{T_V}\ x \to \mathrm{V}) \to (\mathrm{T_V}\ (f\ x\ y) \to \mathrm{V}$

We have:

$\widehat{\mathrm{U}}_{f,g} : \mathrm{V}$

$\mathrm{U}_{f,g} : \mathrm{Set}$

$\mathrm{T_V}\ (\widehat{\mathrm{U}}\ f\ g) = \mathrm{U}\ f\ g$

$\widehat{\mathrm{T}}_{f,g} : \mathrm{U}_{f,g} \to \mathrm{V}$

We define

$\mathrm{T}_{f,g}\ a := \mathrm{T_V}\ (\widehat{\mathrm{T}}_{f,g}\ a)$

Formulation of Mahlo Universe

V closed under universe constructions:

$\widehat{\mathbb{N}}_{\mathrm{V}} : \mathrm{V}$

$\mathrm{T}_{\mathrm{V}}\ \widehat{\mathbb{N}}_{\mathrm{V}} = \mathbb{N}$

$\widehat{\Pi}_{\mathrm{V}} : (a : \mathrm{V}) \to (\mathrm{T}_{\mathrm{V}}\ a \to \mathrm{V}) \to \mathrm{V}$

$\widehat{\mathrm{T}}_{\mathrm{V}}\ (\widehat{\Pi}_{\mathrm{V}}\ a\ b) = (x : \mathrm{T}_{\mathrm{V}}\ a) \to \mathrm{T}_{\mathrm{V}}\ b$

$\mathrm{U}_{f,g}$ closed under universe constructions:

$\widehat{\mathbb{N}}_{f,g} : \mathrm{U}_{f,g}$

$\widehat{\mathrm{T}}_{f,g}\ \widehat{\mathbb{N}}_{f,g} = \widehat{\mathbb{N}}_{\mathrm{V}}$

$\widehat{\Pi}_{f,g} : (a : \mathrm{U}_{f,g}) \to (\mathrm{T}_{f,g}\ a \to \mathrm{U}_{f,g}) \to \mathrm{U}_{f,g}$

$\widehat{\mathrm{T}}_{f,g}\ (\widehat{\Pi}_{f,g}\ a\ b) = \widehat{\Pi}_{\mathrm{V}}\ (\widehat{\mathrm{T}}_{f,g}\ a)\ (\widehat{\mathrm{T}}_{f,g} \circ b)$

Formulation of Mahlo Universe

$\mathrm{U}_{f,g}$ closed under f, g:

$$
\begin{array}{lll}
\widehat{\mathrm{f}} & : & (x : \mathrm{U}_{f,g}) \to (\mathrm{T}_{f,g}\ x \to \mathrm{U}_{f,g}) \to \mathrm{U}_{f,g} \\
\widehat{\mathrm{g}} & : & (x : \mathrm{U}_{f,g}) \\
 & & \to (y : \mathrm{T}_{f,g}\ x \to \mathrm{U}_{f,g}) \\
 & & \to \mathrm{T}_{\mathrm{V}}\ (f\ (\widehat{\mathrm{T}}_{f,g}\ x)\ (\widehat{\mathrm{T}}_{f,g} \circ y)) \\
 & & \to \mathrm{U}_{f,g}
\end{array}
$$

$$
\begin{array}{lll}
\widehat{\mathrm{T}}_{f,g}\ (\widehat{\mathrm{f}}\ a\ b) & = & f\ (\widehat{\mathrm{T}}_{f,g}\ a)\ (\widehat{\mathrm{T}}_{f,g} \circ b) \\
\widehat{\mathrm{T}}_{f,g}\ (\widehat{\mathrm{g}}\ a\ b\ c) & = & g\ (\widehat{\mathrm{T}}_{f,g}\ a)\ (\widehat{\mathrm{T}}_{f,g} \circ b)\ c
\end{array}
$$

Problems of Mahlo Universe

- Elements of V are constructed, depending on total functions

$$f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$$

- This looks impredicative.
- However, for defining U_f, only the restriction of f to $\mathrm{Fam}(\mathrm{U}_f)$ is needed to be total.
- Problem: In Martin-Löf Type Theory all functions are total.
- In Feferman's explicit mathematics we can refer to the collection of all functions.
- There we will define a Mahlo universe which is axiomatized as being completely constructed from below.

Extended Predicative Mahlo Universe

- We write M for the extended predicative Mahlo universe.
- We can encode families of sets into sets therefore we need only to consider functions $f : \mathrm{M} \to \mathrm{M}$ rather than $f : \mathrm{Fam}(\mathrm{M}) \to \mathrm{Fam}(\mathrm{M})$.

Extended Predicative Mahlo Universe

- Let M a universe in explicit mathematics, i.e.
  $\mathcal{U}(\mathrm{M})$.
- Define for a, f, v such that $\Re_{\Re}(v)$ the pre universe closed under universe constructions, a, f relative to v:

$$\Re_{\Re}(\mathrm{pre}\ (a, f, v))$$

Closure of pre (a,f,v)

- pre (a, f, v) is an as closed as possible subuniverse of v closed as much as possible under a, v:
- pre (a, f, v) is closed under universe constructions, provided result is in v.
  - Closure under nat:

$$\mathrm{nat}\ \dot{\in}\ v \to \mathrm{nat}\ \dot{\in}\ \mathrm{pre}\ (a, f, v)$$

  - Closure under co:

$$(a\ \dot{\in}\ \mathrm{pre}\ (a, f, v) \wedge \mathrm{co}\ a\ \dot{\in}\ v) \to \mathrm{co}\ a\ \dot{\in}\ \mathrm{pre}\ (a, f, v)$$

  - Closure under j:

$$(x\ \dot{\in}\ \mathrm{pre}\ (a, f, v) \wedge y : x\ \dot{\to}\ \mathrm{pre}\ (a, f, v) \wedge \mathrm{j}\ (x, y)\ \dot{\in}\ v)$$
$$\to \mathrm{j}\ (x, y) \in \mathrm{pre}\ (a, f, v)$$

Closure of pre (a,f,v)

- pre (a, f, v) is closed under a, f provided result is in v.
  - Closure under a:

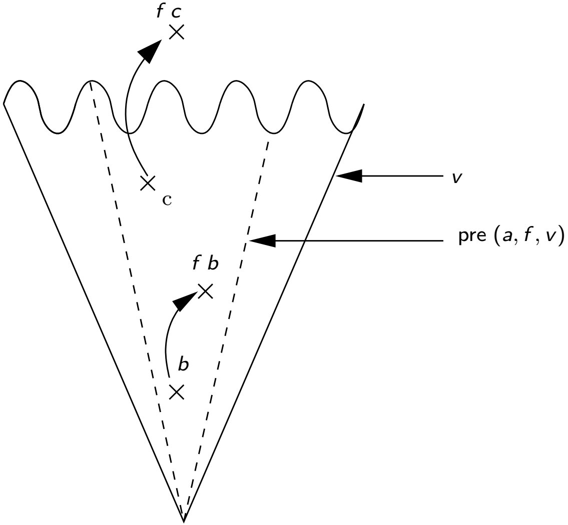
$$a \mathrel{\dot{\in}} v \rightarrow a \mathrel{\dot{\in}} \text{pre}\,(a, f, v)$$

  - Closure under f:

$$(a \in \text{pre}\,(a, f, v) \wedge f\ a \in v) \rightarrow f\ a \in \text{pre}\,(a, f, v)$$

‘

pre (a,f,v)



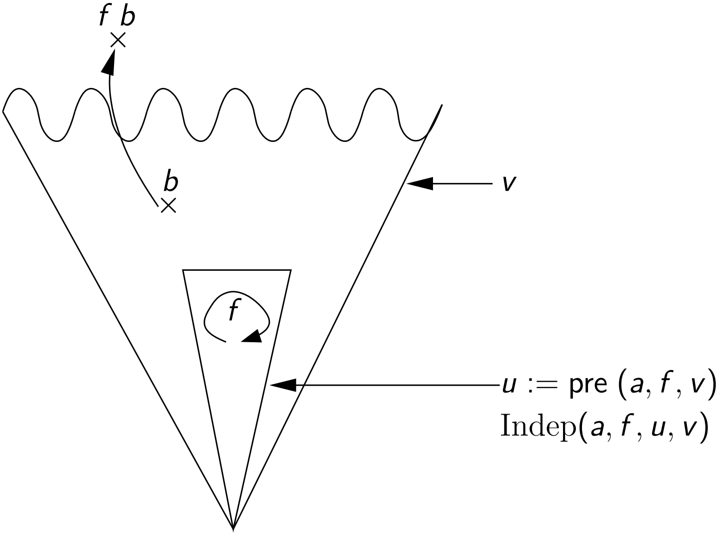
Independence of u from v relative to a,f

- We define that a set u is independent of v relative to a, f, written as

$$\mathrm{Indep}(a, f, u, v)$$

iff all elements we can construct from u using a, f, or universe constructions are already in v
(Therefore if $u = \mathrm{pre}(a, f, v)$, then they will be added to u).
  - $\mathrm{nat} \mathrel{\dot{\in}} v$.
  - $x \mathrel{\dot{\in}} u \rightarrow \mathrm{co}\ x \mathrel{\dot{\in}} v$.
  - $(x \mathrel{\dot{\in}} u \wedge (y : x \mathrel{\dot{\rightarrow}} u)) \rightarrow \mathrm{j}\ (x, y) \mathrel{\dot{\in}} v$.
  - $a \in v$,
  - $x \mathrel{\dot{\in}} u \rightarrow f\ x \mathrel{\dot{\in}} v$.

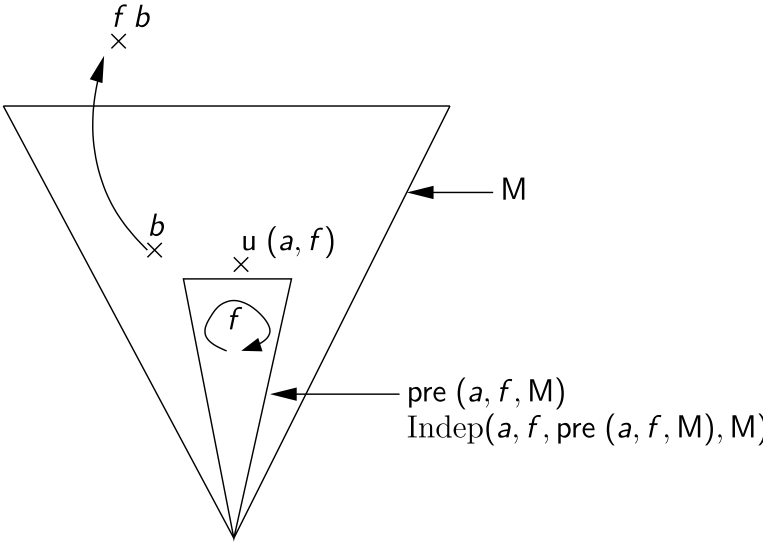
Indpt(a,f,pre(a,f,v),v)



Main Introduction Rule for M

- $$\forall f.\,\mathrm{Indep}(a, f, \mathsf{pre}\,(a, f, \mathsf{M}), \mathsf{M}) \to (\Re(\mathsf{u}\,(a, f)) \land \mathsf{u}\,(a, f) =_{\mathrm{ext}} \mathsf{pre}\,(a, f; \mathsf{M}) \land \mathsf{u}\,(a, f) \in \mathsf{M}) \quad .$$

Introduction Rule for M



Interpretation of Axiomatic Mahlo

- One can easily show:

$$(a \mathrel{\dot\in} \mathrm{M} \wedge f : \mathrm{M} \mathrel{\dot\to} \mathrm{M}) \to \mathrm{Indep}(a, f, \mathrm{pre}\ (a, f, \mathrm{M}), \mathrm{M})$$

therefore

$$\begin{aligned}(a \mathrel{\dot\in} \mathrm{M} \wedge f : \mathrm{M} \mathrel{\dot\to} \mathrm{M}) \to (&\mathrm{u}\ (a, f) \mathrel{\dot\in} \mathrm{M} \\ &\wedge \mathcal{U}(\mathrm{u}\ (a, f)) \\ &\wedge\ a \mathrel{\dot\in} \mathrm{u}(a, f) \\ &\wedge\ f : \mathrm{u}(a, f) \to \mathrm{u}(a, f))\end{aligned}$$

- So M closed under axiomatic Mahlo constructions.
- Therefore extended predicative Mahlo has at least strength of axiomatic Mahlo.

Main Idea

- We define by recursion over ordinal a ternary predicate P^α, from which one obtains approximations

$$\mathrm{R}^\alpha, \in^\alpha, \notin^\alpha$$

- Done by adding to P clauses for all set constructions we can define up to now.

Encoding of Re, $\in$, $\notin$ as one Ternary Predicate

- For P a ternary predicate we define

$$\begin{array}{lcl} \Re_P(a) & := & P(a,0,0) \\ a \in_P b & := & P(a,b,1) \\ a \notin_P b & := & P(a,b,2) \end{array}$$

- Define $\mathrm{Pred}(\Re, \in, \notin)$ to be the ternary predicate encoding $\Re$, $\in$, $\notin$.
- We define operators $\mathcal{A}$ from ternary predicates to ternary predicates.
- We define

$$\begin{array}{lcl} \Re^\alpha & := & \Re_{\mathcal{A}^\alpha}, \\ \in^\alpha & := & \in_{\mathcal{A}^\alpha}, \\ \notin^\alpha & := & \notin_{\mathcal{A}^\alpha}, \end{array}$$

Basic Setup

- Constants are interpreted as natural numbers.
- Application is interpreted as $\{s\}(t)$.
- We define encodings for the terms occurring such as

$$\widehat{\mathrm{nat}},\ \widehat{\mathrm{suc}}(a),\ \widehat{\mathrm{co}}(a),\ \widehat{\mathrm{j}}(a,f)$$

- encode

$$(a,b) \text{ by } \langle a,b\rangle$$

- define codes

$$\ulcorner \mathrm{nat} \urcorner,\ \ulcorner \mathrm{co} \urcorner,\ \ulcorner \mathrm{j} \urcorner, \ldots$$

such that for instance

$$\{\ulcorner \mathrm{j} \urcorner\}(\langle a,f\rangle) = \widehat{\mathrm{j}}(a,f)$$

Basic Universe Constructions

The basic universe constructions are added in one step:

- Closure under nat:

$$
\begin{array}{lcl}
\mathfrak{R}^{\mathrm{nat}}(a) & := & a = \widehat{\mathrm{nat}} \\
b \in^{\mathrm{nat}} a & := & a = \widehat{\mathrm{nat}} \wedge \exists n.b = \widehat{\mathrm{suc}}^{n}\ \widehat{\mathrm{zero}} \\
b \notin^{\mathrm{nat}} a & := & a = \widehat{\mathrm{nat}} \wedge \forall n.b \neq \widehat{\mathrm{suc}}^{n}\ \widehat{\mathrm{zero}} \\
\mathcal{A}^{\mathrm{nat}}(P) & := & \mathrm{Pred}(\mathfrak{R}^{\mathrm{nat}}, \in^{\mathrm{nat}}, \notin^{\mathrm{nat}})
\end{array}
$$

Basic Universe Constructions

- Clousure under co:

$$
\begin{array}{lcl}
\Re_P^{\mathrm{co}}(a,u) & := & a = \widehat{\mathrm{co}}(u) \wedge \Re_P(u) \\
\Re_P^{\mathrm{co}}(a) & := & \exists u.\Re_P^{\mathrm{co}}(a,u) \\
b \in_P^{\mathrm{co}} a & := & \exists u.\Re_P^{\mathrm{co}}(a,u) \wedge b \notin_P a \\
b \notin_P^{\mathrm{co}} a & := & \exists u.\Re_P^{\mathrm{co}}(a,u) \wedge b \in_P a \\
\mathcal{A}^{\mathrm{co}}(P) & := & \mathrm{Pred}(\Re_P^{\mathrm{co}}, \in_P^{\mathrm{co}}, \notin_P^{\mathrm{co}})
\end{array}
$$

Basic Universe Constructions

- Closure under j:

$$
\begin{array}{lcl}
\mathfrak{R}^{\mathrm{j}}_P(a,u,f) & := & a = \widehat{\mathrm{j}}(u,f) \wedge \mathfrak{R}_P(u) \wedge \forall x.x \notin_P u. \vee \mathfrak{R}_P(\{f\}(x)) \\
\mathfrak{R}^{\mathrm{j}}_P(a) & := & \exists u,f.\mathfrak{R}^{\mathrm{j}}_P(a,u,f) \\
b \in^{\mathrm{j}}_P a & := & \exists u,f.\mathfrak{R}^{\mathrm{j}}_P(a,u,f) \\
 & & \wedge \exists y,z.b = \langle y,z\rangle \wedge y \in_P u \wedge z \in_P \{f\}(y) \\
b \in^{\mathrm{j}}_P a & := & \exists u,f.\mathfrak{R}^{\mathrm{j}}_P(a,u,f) \wedge \\
 & & \forall y,z.b = \langle y,z\rangle \to y \notin_P u \vee z \notin_P \{f\}(y) \\
\mathcal{A}^{\mathrm{j}}(P) & := & \mathrm{Pred}(\mathfrak{R}^{\mathrm{j}}_P, \in^{\mathrm{j}}_P, \notin^{\mathrm{j}}_P)
\end{array}
$$

Inductive Generation

- Inductive generation will be defined by adding first the elements iteratively until all elements are there, and then adding the code to $\Re^{\alpha}$:

$$
\begin{array}{lcl}
\Re_P^{\mathrm{pre-i}}(a,u,v) & := & a = \widehat{\mathrm{i}}(u,v) \wedge \Re_P(u) \wedge \Re_P(v) \\
b \in_P^{\mathrm{i}} a & := & \exists u,v.\Re_P^{\mathrm{pre-i}}(a,u,v) \wedge b \in_P u \\
 & & \quad \wedge \forall x.x \notin_P u \vee \langle x,b\rangle \notin_P v \vee x \in_P a \\
\mathrm{Clos}_P^{\mathrm{i}}(u,v) & := & \forall x \in_P^{\mathrm{i}} \widehat{\mathrm{i}}(u,v).x \in_P \widehat{\mathrm{i}}(u,v) \\
\Re_P^{\mathrm{i}}(a,u,v) & := & \Re_P^{\mathrm{pre-i}}(a,u,v) \wedge \mathrm{Clos}_P^{\mathrm{i}}(u,v) \\
\Re_P^{\mathrm{i},+}(a) & := & \exists u,v.\Re_P^{\mathrm{i}}(a,u,v) \\
b \notin_P^{\mathrm{i}} a & := & \exists u,v.\Re_P^{\mathrm{i}}(a,u,v) \wedge b \notin_P a
\end{array}
$$

The union of all unierse constructions

$$\begin{array}{lcl} \Gamma^{\text{univ}}_{P}(u', x) & := & \Re^{\text{nat}}(x) \vee \exists u, v \in_P u'.\exists f. \\ & & \quad \Re^{\text{co}}_{P}(x, u) \vee \cdots \end{array}$$

$$\begin{array}{lcl} \Re^{\text{univ}}_{P}(x) & := & \Re^{\text{nat}}(x) \vee \exists u, v, f.\Re^{\text{co}}_{P}(x, u) \vee \cdots \\ a \in^{\text{univ}}_{P} b & := & a \in^{\text{nat}}_{P} b \vee b \in^{\text{co}}_{P} a \vee \cdots \\ a \notin^{\text{univ}}_{P} b & := & a \notin^{\text{nat}}_{P} b \wedge \wedge b \notin^{\text{co}}_{P} a \wedge \cdots \end{array}$$

Preuniverses

$$
\begin{array}{lcl}
\Re_P^{\mathrm{pre-pre}}(a', a, f, v) & := & a' = \widehat{\mathrm{pre}}(a, f, v) \\
b \in_P^{\mathrm{pre,pot}} a' & := & \exists a, f, v. \Re_P^{\mathrm{pre-pre}}(a', a, f, v) \\
 & & \quad \wedge (b = a \vee (\exists x \in_P a'. b \simeq \{f\}(x)) \\
 & & \quad\quad \vee \Gamma_P^{\mathrm{univ}}(\widehat{\mathrm{pre}}(a, f, v), b)) \\
b \in_P^{\mathrm{pre}} a' & := & b \in_P^{\mathrm{pre,pot}} a' \wedge b \in_P v \\
\mathrm{Clos}_P^{\mathrm{pre}}(a', v) & := & \exists a, f. a' = \widehat{\mathrm{pre}}(a, f, v) \wedge \forall b \in_P^{\mathrm{pre}} a'. b \in_P a' \\
 & & \\
\mathrm{Indep}^{\mathrm{pre}}(a', v) & := & \exists a, f. \Re_P^{\mathrm{pre}}(a', a, f, v) \wedge \forall b \in_P^{\mathrm{pre,pot}} a'. b \in_P v \\
 & & \\
\Re_P^{\mathrm{pre}}(a', a, f, v) & := & \Re_P^{\mathrm{pre-pre}}(a', a, f, v) \\
 & & \wedge((\mathrm{Clos}_P^{\mathrm{pre}}(a', v) \wedge (\Re_P(v) \vee \mathrm{Indep}^{\mathrm{pre}}(a', v))))
\end{array}
$$

- Even if v is not fixed by becoming a name, we can introduce $\widehat{\mathrm{pre}}(a, f, v)$, if it is independent of v.

Preuniverses

$$\begin{array}{lcl}\Re_P^{\mathrm{pre},+}(a') & := & \exists a, f, v.\Re_P^{\mathrm{pre}}(a', a, f, v) \\ b' \not\in_P^{\mathrm{pre}} a' & := & \exists a, f, v.\Re_P^{\mathrm{pre}}(a' a, f, v) \wedge \neg(b' \in_P a')\end{array}$$

Addition of u(a,f)

$$
\begin{array}{lll}
\Re^{\mathrm{u,pot}}_P(a', a, f) & := & a' = \widehat{\mathrm{u}}(a, f) \land \mathrm{Indep}^{\mathrm{pre}}(\widehat{\mathrm{pre}}(a, f), \mathrm{M}) \\
\Re^{\mathrm{u}}_P(a', a, f) & := & a' = \widehat{\mathrm{u}}(a, f) \land \mathrm{Clos}^{\mathrm{pre}}(\widehat{\mathrm{pre}}(a, f)) \land \mathrm{Indep}^{\mathrm{pre}}(\widehat{\mathrm{pre}}(a, f), \mathrm{M}) \\
\Re^{\mathrm{u},+}_P(a') & := & \exists a, f . \Re^{\mathrm{u}}_P(a', a, f) \\
b \in^{\mathrm{u}}_P a' & := & \exists a, f . \Re^{\mathrm{u}}_P(a', a, f) \\
 & & \quad \land b \in_P \widehat{\mathrm{pre}}(a, f, \widehat{\mathrm{M}}) \\
b \notin^{\mathrm{u}}_P a' & := & \exists a, f . \Re^{\mathrm{u}}_P(a', a, f) \\
 & & \quad \land b \notin_P \widehat{\mathrm{pre}}(a, f, \widehat{\mathrm{M}})
\end{array}
$$

M

$$
\begin{array}{lcl}
\Re_P^{\mathrm{pre-M}}(a) & := & a = \widehat{\mathrm{M}} \\
b \in_P^{\mathrm{M,pot}} a & := & \Re_P^{\mathrm{pre-M}}(a) \\
 & & \wedge(\Gamma_P^{\mathrm{univ}}(\widehat{\mathrm{M}}, b) \vee \Gamma_P^{\mathrm{i,pot}}(\widehat{\mathrm{M}}, b) \vee \exists a, f.\Re_P^{\mathrm{u,pot}}(b, a, f)) \\
b \in_P^{\mathrm{M}} a & := & \Re_P^{\mathrm{pre-M}}(a) \\
 & & \wedge(\Gamma_P^{\mathrm{univ}}(\widehat{\mathrm{M}}, b) \vee \Gamma^{\mathrm{i}}{}_P(\widehat{\mathrm{M}}, b) \vee \exists a, f.\Re_P^{\mathrm{u}}(b, a, f)) \\
\mathrm{Clos}_P^{\mathrm{M}} & := & \forall b \in_P^{\mathrm{M,pot}} \widehat{\mathrm{M}}.b \in_P \widehat{\mathrm{M}} \\
\Re_P^{\mathrm{M}}(a) & := & \Re_P^{\mathrm{pre-M}}(a) \wedge \mathrm{Clos}_P^{\mathrm{M}} \\
b \notin_P^{\mathrm{M}} a & := & \Re_P^{\mathrm{M}}(a) \wedge \neg(b \in_P \widehat{\mathrm{M}})
\end{array}
$$

Final Operator

- We form one operator $\mathcal{A}$ for closure under all constructions for forming sets.
- Finding suitable monotonicity conditions is a bit tricky.
- Closes at the first recursively hyperinaccessible above the first recursively Mahlo ordinal (work in progress, it might be possible to improve the bound).
- Therefore Kripke-Platek set theory extended by axioms expressing that the collection of ordinals has this properties proves the consistency of extended predicative Mahlo.

Conclusion

- Introduction of Martin-Löf Type Theory and Explicit Mathematics in parallel.
- For obtaining a fully satisfying theory of the Mahlo universe we need the fact that Explicit Mathematics has access to the collection of terms and partial application.
- In Explicit Mathematics the Mahlo universe can be constructed from below.
- It would be nice to develop Explicit Mathematics in full so that it can serve as a full alternative to Martin-Löf Type Theory.
- Use of Explicit Mathematics for developing interactive theorem provers?