Consistency, Physics and Coinduction

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Consistency, Gödel's Incompleteness Theorem, and Physics

Coinduction

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Coinduction

Uncertainty in Mathematics

- We have a proof of Fermat's last Theorem, by now thoroughly checked.
- We can't exclude that there is a counter example.
  - Reason: By Gödel's Incompleteness Theorem we cannot exclude that axiomatization of mathematics used is consistent.
- A counter example could exist, and would imply that the axiomatization used is inconsistent.
- Although this uncertainty is well known, it is not discussed openly.
  - Almost as if we were hiding the truth.
- Different in physics – physicists are proud of the limitation of physics (e.g. limit of speed of light, Heisenberg's uncertainty principle).

Comparison with Physics

- This lack of absolute certainty is similar to the situation in physics.
- The laws of physics cannot be tested completely.
- We cannot exclude that in other parts of the universe different laws of physics hold.
    - They only need to be in such a way that they appear to us as if they were following the laws of physics as we know them on our planet.
- Because of the lack of a unifying theory we know that the laws of physics are incorrect.
- Laws of physics had to be changed several times in history (relativity theory, quantum mechanics, string theory?).

Effects of Changes of Laws in Physics

- When the laws of physics had to be changed, they didn't affect most calculations done before.
  - Results were thoroughly checked through experiments, so these results are still unaffected.
  - Effects happened only in extreme cases (high speed, small distances). In ordinary life we don't notice the effects of quantum mechanics or relativity theory.

Effects of a Potential Inconsistency in Mathematics

- Reverse mathematics has shown hat most mathematical theorems use very little proof theoretic strength.
  - If there were an inconsistency, it would most likely affect proof theoretically very strong theories.
  - Most mathematical theorems would not be affected.
- In fact as in physics mathematical axioms have been thoroughly "tested".
  - If there were an inconsistency, it must be very involved and would probably not have been used in most mathematical proofs.

Experiments in Physics

- In Physics experiments are used in order to obtain a high degree of certainty.
  - They will never provide absolute certainty.

Experiments in Mathematics

- In Logic lots of "experiments" are carried out as well.
- Simplest form is searching for an inconsistency.
- More involved "experiments are:
  - **Proof theoretic analysis**:
    Reduction of the consistency of mathematical theories to the well-foundedness of an ordinal notation system.
  - **Normalisation proofs.**
  - **Type theoretic foundations:**
    Proof of the consistency of a mathematical theory in a type theory together with some philosophical insight into its consistency (meaning explanations.
  - **Modelling** of one theory in another.
  - **Reverse mathematics**.
  - Lots of **other meta-mathematical investigations**.

Certainty in Mathematics

- No meta-mathematical investigation, even in combination with philosophical investigations, can get around Gödel's Incompleteness Theorem.
- Therefore we **cannot obtain absolute certainty**.
- However we can consider them as experiments and get a certainty similar to what we have in physics.

Conclusion (Part 1)

- Mathematics can be seen as an Empirical Science.
- Mathematics tries to determine laws of the infinite and derive conclusions from those laws.
- We form models of the infinite (axiom systems).
- We carry out experiments.
- We have obtained a high degree of certainty, but will never obtain absolute certainty.
- If an inconsistency were found it probably wouldn't have a huge direct impact on the results obtained in mathematics.

Consistency, Gödel's Incompleteness Theorem, and Physics

Coinduction

Lists

- We assume
  - a set of terms Term formed from
    - constructors
    - variables,
    - function symbols,
    - λ-abstraction
  - together with confluent reduction rules for terms starting with a function symbol.
- Equality on terms is the equivalence relation generated from

$$(s \longrightarrow s) \Rightarrow (s = t)$$

- We identify terms which are equal.
- The set of lists is defined as

$$\mathrm{List} := \bigcap \{X \subseteq \mathrm{Term} \mid \mathrm{nil} \in X \wedge \\ \forall n \in \mathbb{N}. \forall a \in X. \mathrm{cons}(n, a) \in X\}$$

Example Proof using the Definition of List

- Assume function symbol append together with reduction rules

$$\begin{array}{lcl} \mathrm{append}(\mathrm{nil}, l) & \longrightarrow & l \\ \mathrm{append}(\mathrm{cons}(n, l), l') & \longrightarrow & \mathrm{cons}(n, \mathrm{append}(l, l')) \end{array}$$

- We show $\forall l \in \mathrm{List}.\mathrm{append}(l, \mathrm{nil}) = l$:
  - $A := \{l \in \mathrm{List} \mid \mathrm{append}(l, \mathrm{nil}) = l\}$.
  - $\mathrm{nil} \in A$, since $\mathrm{append}(\mathrm{nil}, \mathrm{nil}) = \mathrm{nil}$.
  - $\forall n \in \mathbb{N}.\forall l \in A.\mathrm{cons}(n, l) \in A$

    since $\mathrm{append}(\mathrm{cons}(n, l), \mathrm{nil}) = \mathrm{cons}(n, \mathrm{append}(l, \mathrm{nil})) \stackrel{l \in A}{=} \mathrm{cons}(n, l)$.
  - Therefore $\mathrm{List} \subseteq A$.

Proof by Induction

- Principle of induction:
  - Assume $\varphi(\mathrm{nil})$, $\forall n \in \mathbb{N}.\forall l \in \mathrm{List}.\varphi(l) \to \varphi(\mathrm{cons}(n, l))$.
  - Then $\forall l \in \mathrm{List}.\varphi(l)$.
- Follows directly from definition of List.
- Using induction we can proof $\forall l \in \mathrm{List}.\mathrm{append}(l, \mathrm{nil}) = l$:
  - Base case: $\mathrm{append}(\mathrm{nil}, \mathrm{nil}) = \mathrm{nil}$.
  - Induction step: Assume $\mathrm{append}(l, \mathrm{nil}) = l$. Then $\mathrm{append}(\mathrm{cons}(n, l), \mathrm{nil}) = \mathrm{cons}(n, \mathrm{append}(l, \mathrm{nil})) \overset{\mathrm{IH}}{=} \mathrm{cons}(n, l)$.
  - Therefore $\forall l \in \mathrm{List}.\mathrm{append}(l, \mathrm{nil}) = l$.

Comparison of the proofs

- Both proofs are descriptions of the same content.
- Proof by induction is more intuitive.

From Lists to Colists

- Let $F(X) := \{*\} + \mathbb{N} \times X$.
- Define
$$\begin{array}{lcl} \mathrm{nil}' & := & \mathrm{inl}(*) \\ \mathrm{cons}'(n, l) & := & \mathrm{inr}(\langle n, l \rangle) \end{array}$$
- So $F(X) = \{\mathrm{nil}'\} \cup \{\mathrm{cons}'(n, l) \mid n \in \mathbb{N} \wedge l \in X\}$.
- Define
$$\begin{array}{lcl} \multicolumn{3}{l}{\mathrm{intro} : F(\mathrm{List}) \to \mathrm{List}} \\ \mathrm{intro}(\mathrm{nil}') & = & \mathrm{nil} \ , \\ \mathrm{intro}(\mathrm{cons}'(n, l)) & = & \mathrm{cons}(n, l) \ . \end{array}$$
- $$\mathrm{List} = \bigcap \{X \subseteq \mathrm{Term} \mid \forall l \in F(X).\mathrm{intro}(l) \in X\}$$

From Lists to Colists

- Define

$$\text{coList} := \bigcup \{X \subseteq \text{Term} \mid \forall l \in X.\text{case}(l) \in F(X)\}$$

- Example:
  - Assume a function symbol $a \in \text{Term}$, $\text{case}(a) \longrightarrow \text{cons}'(n, a)$.
  - Let $A := \{a\}$.
  - $\forall x \in A.\text{case}(x) \in F(A)$.
  - Therefore $A \subseteq \text{coList}$, $a \in \text{coList}$.

Proof using the Definition of List

- Assume a function symbol f with reduction rules

$$\mathrm{case}(f(n)) \longrightarrow \mathrm{cons}'(n, f(n+1))$$

- Let $A := \{f(n) \mid n \in \mathbb{N}\}$.
- $\forall a \in A.\mathrm{case}(a) \in F(A)$.
- Therefore $A \subseteq \mathrm{coList}$, $\forall n \in \mathbb{N}.f(n) \in \mathrm{coList}$.

Principle of Coinduction

- Assume

$$\forall l.\varphi(l) \to \mathrm{case}(l) = \mathrm{nil}' \lor \\ \exists n \in \mathbb{N}.\exists l' \in \mathrm{Term}.\mathrm{case}(l) = \mathrm{cons}'(n, l') \land \varphi(l')$$

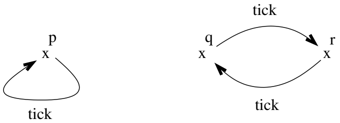
  Then $\forall l \in \mathrm{Term}.\varphi(l) \to l \in \mathrm{coList}$.
- We show $\forall n \in \mathbb{N}.f(n) \in \mathrm{coList}$ by principle of coinduction:
  - Let $n \in \mathbb{N}$.
  - $\mathrm{case}(f(n)) = \mathrm{cons}'(n, f(n+1))$.
  - $n \in \mathbb{N}$ and by co-IH $f(n+1) \in \mathrm{coList}$,
  - Therefore $f(n) \in \mathrm{coList}$.

Comparison of the proofs

- Both proofs are descriptions of the same content.
- Second proof is a much more intuitive.

Bisimulation

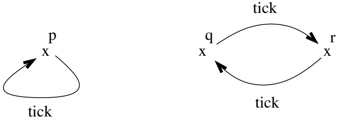
- A labelled transition system is a triple $(P, A, \longrightarrow)$ where P, A are sets and $\longrightarrow \subseteq P \times A \times A$.
  We write $p \stackrel{a}{\longrightarrow} p'$ for $\langle p, a, p' \rangle \in \longrightarrow$.
- Consider the following transition system:



- Bisimulation is given as

$$
\begin{aligned}
\sim := \bigcup \{X \subseteq P \times P \mid (\forall p, q, p' \in P, a \in A. \langle p, q \rangle \in X \wedge p \stackrel{a}{\longrightarrow} p' \\
\rightarrow \exists q' \in P. q \stackrel{a}{\longrightarrow} q' \wedge \langle p', q' \rangle \in X) \\
\wedge \cdots \text{symmetric case} \cdots \}
\end{aligned}
$$

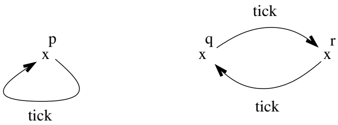
Proof using the Definition of $\sim$



- Let $X := \{\langle p, q\rangle, \langle p, r\rangle\}$.
- Take $\langle p, q\rangle \in X$, and let $p \xrightarrow{a} p'$.

  Then $p' = p$, $a = \text{tick}$, $q \xrightarrow{\text{tick}} r$ and $\langle p, r\rangle \in A$.
- Similarly for other cases.
- Therefore $X \subseteq\sim$, $p \sim q$, $p \sim r$.

Proof by Principle Coinduction



- We show $p \sim q$ and $p \sim r$.
- Let $p \xrightarrow{a} p'$.

  Then $p' = p$, $a = \text{tick}$, $q \xrightarrow{\text{tick}} r$ and by co-IH $p \sim r$.
- Similarly for other cases.

Comparison of the proofs

- Both proofs are descriptions of the same content.
- Second proof is a much more intuitive.

Conclusion (Part 2)

- Principle of induction is well established and makes proofs much easier.
- In theoretical computer science coinductive principles occur frequently.
- In order to get more intuitive easy proofs we need to establish the use of coinduction in a similar way.
  - Proofs by coinduction are the same as those originating from the definition of coinductively defined sets.
  - However proofs by coinduction can be more intuitive and correspond directly to more formal proofs.