The extended predicative Mahlo Universe in Explicit Mathematics – Model Construction

Anton Setzer
Swansea University
Joint Work with Reinhard Kahle, Lisbon

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Goal of the Talk

- Introduce an alternative formalisation of the Mahlo universe in the context of explicit Mathematics.
- Give a model showing consistency.
- Not yet but almost – determine upper bound for the proof theoretic strength of extended predicative Mahlo universe.

Universes in Type Theory

- In type theory a universe is a type the elements of which represent (via a decoding function) types.
- Usually a universe should be closed under basic constructs for forming types.
- Allows to form internal models of type theory.
- Corresponds to large cardinals in set theory or admissibles in
- Universes allow to reach higher proof theoretic strength.
- Martin-Löf introduced a hierarchy of universe

$$\mathrm{U}_0 \overset{\in}{\subseteq} \mathrm{U}_1 \overset{\in}{\subseteq} \cdots$$

- Palmgren introduced the super universe operator, which defines for a every family of types a universe containing those types.
- He added a universe closed under the super universe operator.

Mahlo Universe

- The Mahlo universe in type theory is
  - a universe $\underset{\sim}{\mathrm{V}}$,
  - which has not only subuniverses corresponding to some operators, but subuniverses corresponding to all operators it is closed under:
  - for every universe operator on V,
    - i.e. every $f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$,
  - there exists a universe $\underset{\sim}{\mathrm{U}_f}$ closed under f.

Illustration of the Mahlo Universe

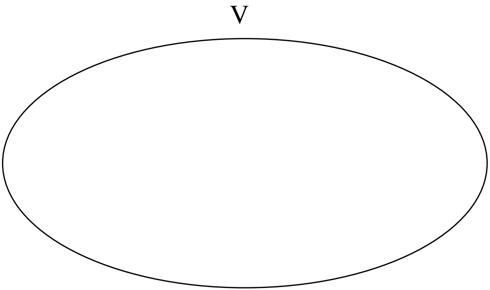
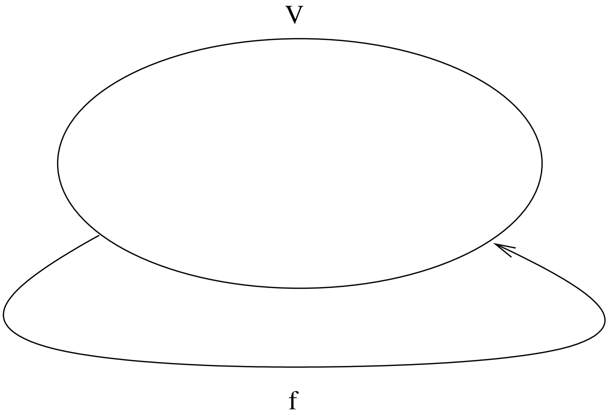


Illustration of the Mahlo Universe



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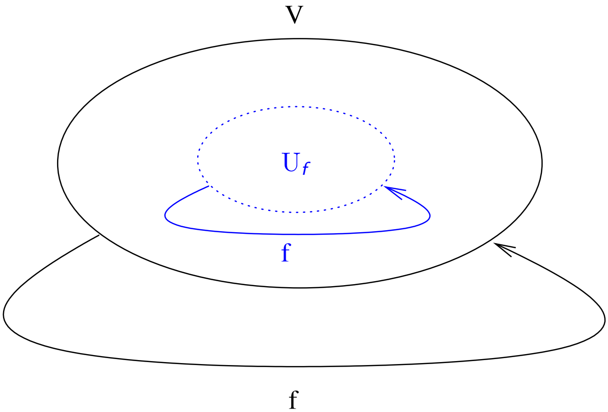
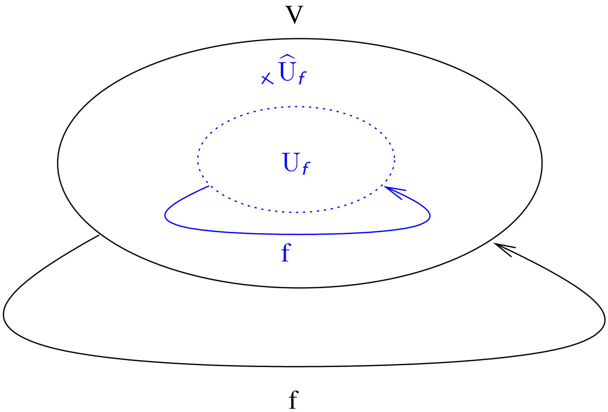


Illustration of the Mahlo Universe



Problem of Mahlo Universe

- Elements of V are constructed, depending on total functions

$$f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$$

- So we introduce elements of V by referring to the totality of V.
- Justification is possible.
- However, many type theoretist doubt the validity of the Mahlo universe as a foundation of mathematics.

Problem of Mahlo Universe

- This reference to the totality of the functions, and therefore impredicativity, can be avoided by observing that for defining U_f, only the restriction of f to $\mathrm{Fam}(\mathrm{U}_f)$ is needed to be total.
- However, in order to make sense of U_f which are not total, we need to refer to arbitrary **partial** functions f.
  - For partial functions f we construct U_f.
  - If we can complete it, i.e. f is total, then we add $\widehat{\mathrm{U}}_f$ to V.
- Problem: In Martin-Löf Type Theory all functions are total.
- We use therefore Feferman's explicit mathematics where we have access to the collection of untyped programs.

Explicit Mathematics

- Framework introduced by Solomon Feferman in order to formalise constructive mathematics.
  - Later classical logic was added.
- It is an untyped alternative to type theory.
- It is presented as a second order language, where first order objects (individuals) can be considered as programs or terms.
- Second order quantifiers range over sets which have names, where names are specific individuals.

Explicit Mathematics

- We can avoid the from a foundational point of view problematic 2nd order language by rewriting it in a first order setting by having a relation

$$\Re(s)$$

selecting individuals which are names for sets and a relation

$$s \mathrel{\dot{\in}} t$$

together with

$$s \mathrel{\dot{\in}} t \to \Re(t)$$

Inductive Generation

- Inductive generation in Explicit Mathematics defines the accessible part of a relation.
- It plays the rôle of the W-type in type theory.
- It is axiomatised as follows:
  Define

$$
\begin{array}{lcl}
x \prec_b y & := & (x, y) \dot{\in} b \\
\mathcal{C}l_{\mathsf{i}}(a, b, c) & := & \forall x \dot{\in} a.(\forall y \dot{\in} a. y \prec_b x \to y \dot{\in} c) \to x \dot{\in} c
\end{array}
$$

Assume $\Re(a) \ \wedge \ \Re(b)$ Then

$\Re(\mathsf{i}(a, b))$

$\mathcal{C}l_{\mathsf{i}}(a, b, \mathsf{i}(a, b))$

$\mathcal{C}l_{\mathsf{i}}(a, b, \varphi) \to \forall x \dot{\in} \mathsf{i}(a, b).\varphi(x)$

Universes

- Universes are names, the elements of which are names, and which are closed under the standard constructs of explicit mathematics for forming sets:

$$
\begin{array}{lll}
\multicolumn{3}{l}{a \in \Gamma_{\mathsf{univ}}(u) :=} \\
 & a = \mathsf{nat} & \text{(natural numbers)} \\
\vee & a = \mathrm{id}, & \text{(identity)} \\
\vee & (\exists x \dot{\in} u. a = \mathsf{co}\, x) & \text{(complement)} \\
\vee & (\exists x, y \dot{\in} u. a = \mathsf{int}\,(x, y)) & \text{(intersection)} \\
\vee & (\exists x \dot{\in} u. a = \mathsf{dom}\, x) & \text{(domain)} \\
\vee & (\exists f. \exists x \dot{\in} u. a = \mathsf{inv}\,(f, x)) & (f^{-1}(x)) \\
\vee & (\exists x \dot{\in} u. \exists f \dot{\in} (x \to u). a = \mathsf{j}\,(x, f)) & (\Sigma\text{-type})
\end{array}
$$

- Now define with

$$\Re_{\Re}(u) := \Re(u) \ \wedge \ (\forall x \dot{\in} u. \Re(x))$$

$$\mathcal{U}(u) := \Re_{\Re}(u) \ \wedge \ \forall x \in \Gamma_{\mathsf{univ}}(u) \to x \dot{\in} u$$

Pre-Universes

- We want to formalise the idea of
  - Whenever we can form a subuniverse U_f closed under f of the Mahlo universe, then we have an element $\widehat{\mathrm{U}}_f$ of the Mahlo universe representing U_f.
- In explicit mathematics traditionally the subuniverses of the Mahlo universe have an additional parameter a, determining an element contained in it.
- In order to formalise this idea we first formulate the notion of a pre universe $\mathrm{pu}\,(a, f, v)$.
- $\mathrm{pu}\,(a, f, v)$ is a universe containing a and closed under f, provided the elements created are in v.
- So it is a subuniverse of v.

Pre-Universes

- We first define the set of potential elements of the pre universe:

$$x \in \Gamma^{\text{pot}}_{\text{pu}}(a, f, u) \quad := \quad x \in \Gamma_{\text{univ}}(u) \ \vee \ x = a \ \vee \ \exists y \,\dot{\in}\, u.x = f\ y$$

- The closure property of a pre universe is now:

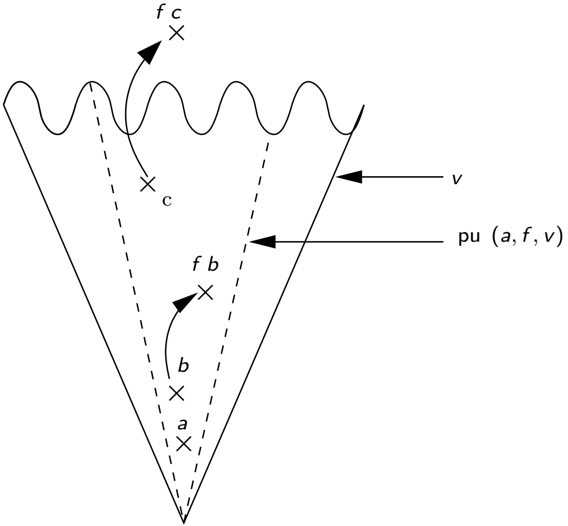
$$\mathcal{C}l_{\text{pu}}(a, f, u, v) \quad := \quad \forall x \in \Gamma^{\text{pot}}_{\text{pu}}(a, f, u) \ \wedge \ x \,\dot{\in}\, v \rightarrow x \,\dot{\in}\, u$$

- Now we formalise that $\text{pu}(a, f, v)$ is the least pre-universe closed under a, f relative to v:
  Assume $\Re_{\Re}(v)$:

$$\begin{array}{l} \mathcal{C}l_{\text{pu}}(a, f, \text{pu}(a, f, v), v) \\ \mathcal{C}l_{\text{pu}}(a, f, \varphi, v) \rightarrow \forall x \,\dot{\in}\, \text{pu}(a, f, v).\varphi(x) \end{array}$$

  We will only need the pre universe for v being the Mahlo universe.

pu (a,f,v)



Indep(a,f,u,v)

- That we have a "subuniverse closed under a and f" corresponds now to the fact that the pre universe is independent of v:
  - The condition that the elements we add to the pre-universe need to be in v is always fulfilled, i.e. the pre universe is independent of v:

$$\mathrm{Indep}(a, f, u, v) \quad := \quad \forall x \in \Gamma_{\mathsf{pu}}^{\mathsf{pot}}(a, f, u).x \mathrel{\dot{\in}} v$$

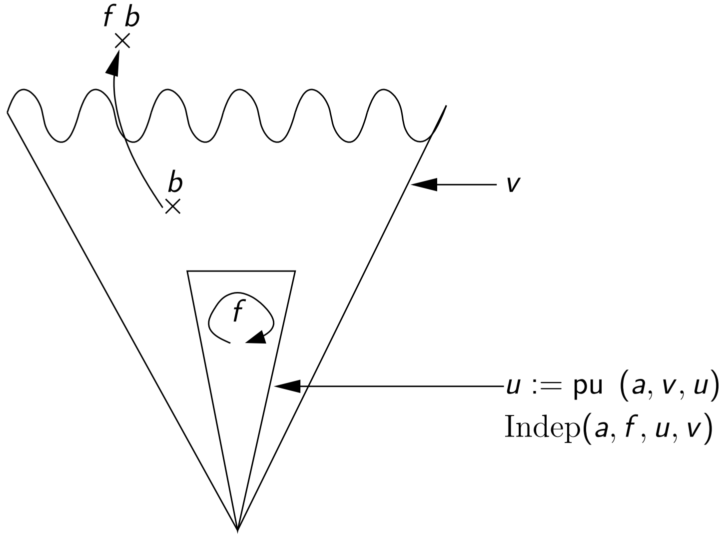
  - Once we have

$$\mathrm{Indep}(a, f, \mathsf{pu}\ (a, f, \mathrm{M}), \mathrm{M})$$

    we know that pu (a, f, M) does not depend on future elements of M introduced.
  - We get immediately

$$\mathrm{Indep}(a, f, \mathsf{pu}\ (a, f, v), v) \to \forall x \in \Gamma_{\mathsf{pu}}^{\mathsf{pot}}(a, f, u, x).x \mathrel{\dot{\in}} \mathsf{pu}\ (a, f, v)$$

Indep(a,f,u,v)

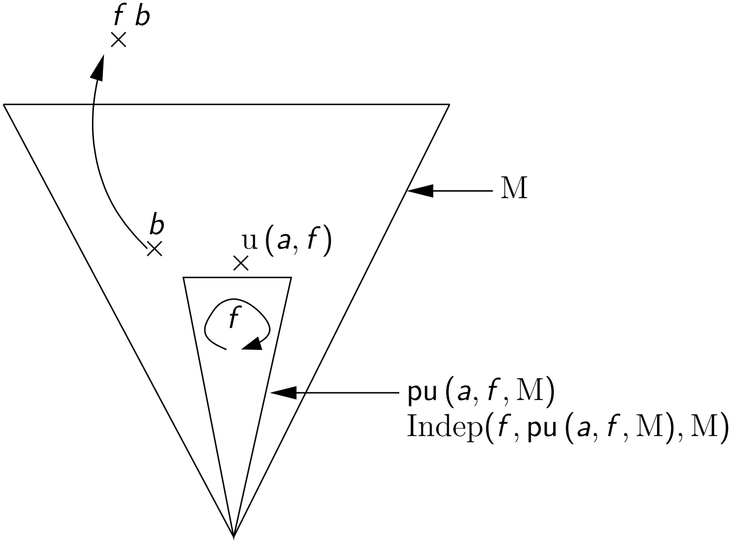


The Extended Predicative Mahlo Universe

- The extended predicative Mahlo universe is a universe, closed under inductive generation, such that for a, f we have
  - If $\mathsf{pu}\,(a, f, \mathrm{M})$ is independent of M, then we have an element $\mathrm{u}\,(a, f)$ of M representing $\mathsf{pu}\,(a, f, \mathrm{M})$.

$$
\begin{array}{l}
\mathcal{U}(\mathrm{M}) \\
\forall a, b \mathrel{\dot\in} \mathrm{M}.\mathsf{i}\ \ (a, b) \in \mathrm{M} \\
\forall a, f.\mathrm{Indep}(a, f, \mathsf{pu}\,(a, f, \mathrm{M}), \mathrm{M}) \to \Re(\mathrm{u}\,(a, f)) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge\ \mathrm{u}\,(a, f) \mathrel{\dot\in} \mathrm{M} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge\ \mathrm{u}\,(a, f) \doteq \mathsf{pu}\,(a, f, \mathrm{M}) \ \ .
\end{array}
$$

The Extended Predicative Mahlo Universe



The Least Mahlo Universe

We can formulate the notion of a least Mahlo universe as follows:

$$
\begin{array}{l}
\mathcal{U}(u) \wedge \\
(\forall a, b \mathrel{\dot{\in}} u.\mathrm{i}\,(a,b) \mathrel{\dot{\in}} u) \wedge \\
(\forall f, a.\mathrm{Indep}(a, f, \mathrm{pu}\,(a, f, u), u) \rightarrow \mathrm{u}\,(a, f) \mathrel{\dot{\in}} u) \\
\rightarrow \mathrm{M} \mathrel{\dot{\subset}} u \quad .
\end{array}
$$

Note that there are no consistent elimination rules for the standard Mahlo universe.

Axiomatic Mahlo in Explicit Mathematics

- Normally by the Mahlo universe in explicit mathematics one means the external Mahlo universe where $\Re$ has the property of Mahloness.
- One can formalise an extended predicative external Mahlo universe as well.
- Tupailo formalised the internal axiomatic Mahlo universe in explicit mathematics as follows:

$$
\begin{array}{l}
\mathcal{U}(\mathrm{M}) \\
\forall a, b \mathrel{\dot{\in}} \mathrm{M}.\mathsf{i}\,(a, b) \mathrel{\dot{\in}} \mathrm{M} \\
a \mathrel{\dot{\in}} \mathrm{M} \ \wedge\ f \in (\mathrm{M} \to \mathrm{M}) \to \mathcal{U}(\mathrm{u}\,(a, f)) \\
\qquad\qquad\qquad\qquad\qquad \wedge\ \mathrm{u}\,(a, f) \mathrel{\dot{\subset}} \mathrm{M}, \\
\qquad\qquad\qquad\qquad\qquad \wedge\ a \mathrel{\dot{\in}} \mathrm{u}\,(a, f) \\
\qquad\qquad\qquad\qquad\qquad \wedge\ f \in (\mathrm{u}\,(a, f) \to \mathrm{u}\,(a, f)) \\
\qquad\qquad\qquad\qquad\qquad \wedge\ \mathrm{u}\,(a, f) \mathrel{\dot{\in}} \mathrm{M}
\end{array}
$$

Extended Predicative Mahlo Universe is a Mahlo Universe

Theorem

The extended predicative Mahlo universe fulfils the axioms of the axiomatic Mahlo universe.

Basic Setup for the Model

- We define the model as subset of $\mathbb{N}^3$.
  If $P \subseteq \mathbb{N}^3$, we interpret $\Re$, $\dot{\in}$, $\dot{\notin}$ as follows:

$$\begin{array}{ll} \Re_P(a) & := P(a,0,0), \\ b \in_P a & := P(a,b,1), \\ b \notin_P a & := P(a,b,2). \end{array}$$

- On the other hand from relations R, $\in'$ $\notin'$ we can define a subset of $\mathbb{N}^3$:

$$\begin{array}{ll} \mathrm{Pred}_{\Re,\in',\notin'}(a,b,c) & := (R(a) \ \wedge \ b = 0 \ \wedge \ c = 0) \vee \\ & \quad\ (a \in' b \ \wedge \ c = 1) \vee \\ & \quad\ (a \notin' b \ \wedge \ c = 2). \\ \mathrm{Pred}(R, \in', \notin') & := \{(a,b,c) \mid \mathrm{Pred}_{R,\in',\notin'}(a,b,c)\} \end{array}$$

- The model will be defined by iterating an operator up to the a large ordinal.

Operator for Defining the Model - j

$$
\begin{array}{lcl}
\Re^{\mathrm{j}}_P(a,u,f) & := & a = \widehat{\mathrm{j}}(u,f) \ \wedge\ \Re_P(u) \wedge \\
 & & (\forall^{\mathrm{pos}} x \in_P u.\exists z.\{f\}(x) \simeq z \ \wedge\ \Re_P(z)) \\
\Re^{\mathrm{j},+}_P(a) & := & \exists u,f.\Re^{\mathrm{j}}_P(a,u,f) \\
b \in^{\mathrm{j}}_P a & := & \exists u,f.\Re^{\mathrm{j}}_P(a,u,f) \ \wedge\ \exists y,z,z'.b = \langle y,z\rangle \ \wedge\ y \in_P u \\
 & & \qquad \wedge\ \{f\}(y) \simeq z' \ \wedge\ z \in_P z' \\
b \notin^{\mathrm{j}}_P a & := & \exists u,f.\Re^{\mathrm{j}}_P(a,u,f) \\
 & & \quad \wedge\ \forall y,z,z'.b = \langle y,z\rangle \\
 & & \qquad \to (y \notin_P u \vee \{f\}(y) \not\simeq z' \vee z \notin_P z') \\
\Gamma^{\mathrm{j}}(P) & := & \mathrm{Pred}(\Re^{\mathrm{j},+}_P, \in^{\mathrm{j}}_P, \notin^{\mathrm{j}}_P)
\end{array}
$$

Operator for Defining the Model - i

$$
\begin{array}{lcl}
\Re_P^{\mathrm{pre-i}}(a,u,v) & := & a = \widehat{\mathrm{i}}(u,v) \ \wedge \ \Re_P(u) \ \wedge \ \Re_P(v) \\
b \in_P^{\mathrm{i}} a & := & \exists u, v. \Re_P^{\mathrm{pre-i}}(a,u,v) \ \wedge \ b \in_P u \\
 & & \quad \wedge \ \forall^{\mathrm{pos}} x \in_P u. \langle x, b \rangle \in_P v \rightarrow^{\mathrm{pos}} x \in_P a \\
\mathcal{C}l_P^{\mathrm{i}}(u,v) & := & \forall x \in_P^{\mathrm{i}} \widehat{\mathrm{i}}(u,v). x \in_P \widehat{\mathrm{i}}(u,v) \\
\Re_P^{\mathrm{i}}(a,u,v) & := & \Re_P^{\mathrm{pre-i}}(a,u,v) \ \wedge \ \mathcal{C}l_P^{\mathrm{i}}(u,v) \\
\Re_P^{\mathrm{i},+}(a) & := & \exists u, v. \Re_P^{\mathrm{i}}(a,u,v) \\
b \notin_P^{\mathrm{i}} a & := & \exists u, v. \Re_P^{\mathrm{i}}(a,u,v) \ \wedge \ b \notin_P a \\
\Gamma^{\mathrm{i}}(P) & := & \mathrm{Pred}(\Re_P^{\mathrm{i},+}, \in_P^{\mathrm{i}}, \notin_P^{\mathrm{i}})
\end{array}
$$

Operator for Defining the Model - pu

$$
\begin{array}{lcl}
\Re_P^{\text{pre}-\text{pu}}(a', a, f, v) & := & a' = \widehat{\text{pu}}(a, f, v) \\
b \in_P^{\text{pre,pot}} a' & := & \exists a, f, v. \Re_P^{\text{pre}-\text{pu}}(a', a, f, v) \\
 & & \quad \wedge (b = a \vee (\exists x \in_P a'. b \simeq \{f\}(x)) \\
 & & \quad\quad \vee \Gamma_P^{\text{univ}}(\widehat{\text{pu}}(a, f, v), b)) \\
b \in_P^{\text{pu}} a' & := & \exists a, f, v. b \in_P^{\text{pu,pot}} a' \wedge \Re_P^{\text{pre}-\text{pu}}(a', a, f, v) \wedge b \in_P v \\
\mathcal{C}l_P^{\text{pu}}(a, f, v) & := & \forall b \in_P^{\text{pu}} \widehat{\text{pu}}(a, f, v). b \in_P \widehat{\text{pu}}(a, f, v) \\
\text{Indep}_P^{\text{pu}}(a', v) & := & \exists a, f. \Re_P^{\text{pu}}(a', a, f, v) \wedge \forall b \in_P^{\text{pu,pot}} a'. b \in_P v \\
\Re_P^{\text{pu,pot}}(a', a, f, v) & := & \Re_P^{\text{pre}-\text{pu}}(a', a, f, v) \\
 & & \wedge (\Re_P(v) \vee \text{Indep}_P^{\text{pu}}(a', v)) \\
\Re_P^{\text{pu}}(a', a, f, v) & := & \Re_P^{\text{pu,pot}}(a', a, f, v) \wedge \mathcal{C}l_P^{\text{pu}}(a, f, v) \\
\Re_P^{\text{pu},+}(a') & := & \exists a, f, v. \Re_P^{\text{pu}}(a', a, f, v) \\
b' \not\in_P^{\text{pu}} a' & := & \exists a, f, v. \Re_P^{\text{pu}}(a' a, f, v) \wedge \neg(b' \in_P a') \\
\Gamma^{\text{pu}}(P) & := & \text{Pred}(\Re_P^{\text{pu},+}, \in_P^{\text{pu}}, \not\in_P^{\text{pu}})
\end{array}
$$

Operator for Defining the Model - u

$$
\begin{array}{lcl}
\Re_P^{\mathsf{u},\mathsf{pre}}(a',a,f) & := & a' = \widehat{\mathsf{u}}(a,f) \\
\Re_P^{\mathsf{u},\mathsf{pot}}(a',a,f) & := & \Re_P^{\mathsf{u},\mathsf{pre}}(a',a,f) \;\wedge\; \mathrm{Indep}_P^{\mathsf{pre}}(\widehat{\mathsf{pu}}(a,f,\mathrm{M}),\mathrm{M}) \\
\Re_P^{\mathsf{u},\mathsf{next}}(a',a,f) & := & \Re_P^{\mathsf{u},\mathsf{pot}}(a',a,f) \;\wedge\; \mathcal{C}l^{\mathsf{pu}}(\widehat{\mathsf{pu}}(a,f,\mathrm{M}),\mathrm{M}) \\
\Re_P^{\mathsf{u}}(a',a,f) & := & \Re_P^{\mathsf{u},\mathsf{next}}(a',a,f) \;\wedge\; \forall x \in_P \widehat{\mathsf{pu}}(a,f,\mathrm{M}).x \in_P \widehat{\mathsf{u}}(a,f) \\
\Re_P^{\mathsf{u},+}(a') & := & \exists a,f.\Re_P^{\mathsf{u}}(a',a,f) \\
b \in_P^{\mathsf{u}} a' & := & \exists a,f.\Re_P^{\mathsf{u},\mathsf{next}}(a',a,f) \\
 & & \qquad \wedge\; b \in_P \widehat{\mathsf{pu}}(a,f,\mathrm{M}) \\
b \notin_P^{\mathsf{u}} a' & := & \exists a,f.\Re_P^{\mathsf{u}}(a',a,f) \\
 & & \qquad \wedge\; b \notin_P \widehat{\mathsf{pu}}(a,f,\mathrm{M}) \\
\Gamma^{\mathsf{u}}(P) & := & \mathsf{Pred}(\Re_P^{\mathsf{u},+}, \in_P^{\mathsf{u}}, \notin_P^{\mathsf{u}})
\end{array}
$$

Operator for Defining the Model - M

$$
\begin{array}{lcl}
\Re_P^{\mathrm{pre-M}}(a) & := & a = \mathrm{M} \\
b \in_P^{\mathrm{M}} a & := & \Re_P^{\mathrm{pre-M}}(a) \\
 & & \wedge\ (\Gamma_P^{\mathrm{univ}}(\mathrm{M}, b) \vee \Gamma^{\mathrm{i}}_P(\mathrm{M}, b) \vee \exists a, f.\Re^{\mathrm{u}}_P(b, a, f)) \\
\mathcal{C}l_P^{\mathrm{M}} & := & \forall b \in_P^{\mathrm{M,pot}} \mathrm{M}.b \in_P \mathrm{M} \\
\Re_P^{\mathrm{M}}(a) & := & \Re_P^{\mathrm{pre-M}}(a)\ \wedge\ \mathcal{C}l_P^{\mathrm{M}} \\
b \notin_P^{\mathrm{M}} a & := & \Re_P^{\mathrm{M}}(a)\ \wedge\ \neg(b \in_P \mathrm{M}) \\
\Gamma^{\mathrm{M}}(P) & := & \mathrm{Pred}(\Re_P^{\mathrm{M}}, \in_P^{\mathrm{M}}, \notin_P^{\mathrm{M}})
\end{array}
$$

Monotonicity of Operators

Definition

1. For P, Q ternary predicates we define

$$
\begin{array}{lll}
P \preceq Q & :\Leftrightarrow & \Re_P \subseteq \Re_Q \\
& & \wedge \forall a, b.(a \in_P b \rightarrow a \in_Q b) \\
& & \qquad \wedge (a \notin_P b \rightarrow a \notin_Q b) \\
& & \qquad \wedge (\Re_P(b) \rightarrow (a \in_P b \leftrightarrow a \in_Q b) \wedge \\
& & \qquad\qquad (a \notin_P b \leftrightarrow a \notin_Q b)
\end{array}
$$

2. Let $\mathcal{B} : \mathcal{P}(\mathbb{N}^3) \rightarrow \mathcal{P}(\mathbb{N}^3)$.
   $\mathcal{B}$ is $\preceq$-monotone iff

$$\forall P, Q.P \preceq Q \rightarrow \mathcal{B}(P) \preceq \mathcal{B}(Q)$$

Monotonicity of Operators

Definition

1. Let $\mathcal{B} : \mathcal{P}(\mathbb{N}^3) \to \mathcal{P}(\mathbb{N}^3)$.
   $\mathcal{B}$ is weakly $\preceq$-monotone iff

$$\forall P, Q.(P \preceq Q \wedge \mathsf{El}_P \restriction \Re_{\mathcal{B}(P)} = \mathsf{El}_Q \restriction R_{\mathcal{B}(P)}) \to \mathcal{B}(P) \preceq \mathcal{B}(Q)$$

2. $\mathcal{B} : \mathcal{P}(\mathbb{N}^3) \to \mathcal{P}(\mathbb{N}^3)$ is a closure operator if $\mathcal{B}$ is weakly monotone and

$$\mathsf{El}_{\mathcal{B}(P)} \restriction (\Re_{\mathcal{B}(P)} \setminus \Re_P) \subseteq \mathsf{El}_P \restriction (\Re_{\mathcal{B}(P)} \setminus \Re_P)$$

3. $\mathcal{B} : \mathcal{P}(\mathbb{N}^3) \to \mathcal{P}(\mathbb{N}^3)$ is inflationary, iff

$$(P \preceq \mathcal{B}(P)) \to (\mathcal{B}(P) \preceq \mathcal{B}^2(P))$$

Lemma

Lemma

1. Γ^{c} *is monotone for each basic set constructor* c.
2. Γ^{i} *is a closure operator.*
3. Γ^{pu} *is a closure operator.*
4. $\Gamma^{\mathsf{u}} \cup \Gamma^{\mathsf{pu}}$ *is a closure operator.*
5. $\Gamma^{\mathsf{i}} \cup \Gamma^{\mathsf{u}} \cup \Gamma^{\mathsf{pu}} \cup \Gamma^{\mathrm{M}}$ *is a closure operator.*

Lemma

The following lemma still needs to be scrutinised:

Lemma

Let Γ *be the overall operator.*

1. *If* κ *is an admissible,* $\Re_{\Gamma^{<\kappa}}(a), \Re_{\Gamma^{<\kappa}}(b)$. *Then* $\Re_{\Gamma^{\kappa}}(\widehat{\mathrm{i}}(a, b))$.
2. *If* κ *is an admissible,* $\mathrm{Indep}^{\mathsf{pre}}_{\Gamma^{<\kappa}}(\widehat{\mathrm{pu}}(a, f, u), u)$, *then* $\Re_{\Gamma^{\kappa}}(\widehat{\mathrm{pu}}(a, f, u))$.
3. *Let* Mahlo *be a recursively Mahlo ordinal*
   If $\mathrm{Indep}^{\mathsf{pre}}_{\Gamma^{<\mathsf{Mahlo}}}(\widehat{\mathrm{pu}}(a, f, \mathrm{M}), \mathrm{M})$, *then* $\Re_{\Gamma^{<\mathsf{Mahlo}}}(\widehat{\mathrm{pu}}(a, f, \mathrm{M}))$,
   $\Re_{\Gamma^{<\mathsf{Mahlo}}}(\widehat{\mathrm{u}}(a, f))$, *and* $\widehat{\mathrm{u}}(a, f) \in_{\Gamma^{<\mathsf{Mahlo}}} \mathrm{M}$.
4. $\mathrm{M} \in \Gamma^{\mathsf{Mahlo}}$.
5. *Let* κ *be a recursively inaccessible ordinal above* Mahlo.
   Then Γ^{κ} *is a model of the extended predicative Mahlo universe.*

Future Work

- Carry out the model in an extension of KPM, should give an upper bound of the proof theoretic strength of the extended predicative Mahlo universe.
  - Upper bound should be KPI + (M).
- Restriction of the model to external extended predicative Mahlo universe.
  - Expected strength should be that of KPM.
- Adapt the well-ordering proof for Type theory + Mahlo universe to obtain a lower bound for the proof theoretic strength.

Martin-Löfisation of the Extended Predicative Mahlo Universe

- Reformulate explicit mathematics and the extended predicative Mahlo universe in a more Martin-Löf style using introduction and elimination rules.
- First step: Formalisation in Agda.