Coinduction, Corecursion, Copatterns

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Algebraic Data Types

In most functional programming languages we have the notion of an algebraic data type, e.g.

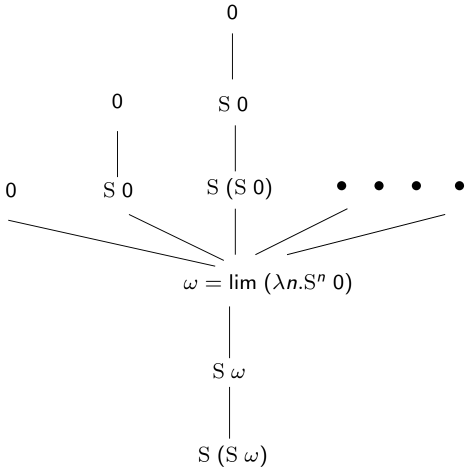
$$
\begin{array}{l}
\mathrm{data}\ \mathbb{N} : \mathrm{Set}\ \ \mathrm{where} \\
\quad \begin{array}{lcl}
\mathsf{0} & : & \mathbb{N} \\
\mathrm{S} & : & \mathbb{N} \to \mathbb{N}
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\mathrm{data}\ \mathrm{NatList} : \mathrm{Set}\ \ \mathrm{where} \\
\quad \begin{array}{lcl}
\mathrm{nil} & : & \mathrm{NatList} \\
\mathrm{cons} & : & (\mathbb{N} \times \mathrm{NatList}) \to \mathrm{NatList}
\end{array}
\end{array}
$$

Algebraic Data Types

$$
\begin{array}{l}
\mathrm{data\ KleeneO : Set\ \ where} \\
\quad \begin{array}{lcl}
\mathsf{0} & : & \mathrm{KleeneO} \\
\mathrm{S} & : & \mathrm{KleeneO} \to \mathrm{KleeneO} \\
\mathsf{lim} & : & (\mathbb{N} \to \mathrm{KleeneO}) \to \mathrm{KleeneO}
\end{array}
\end{array}
$$

Kleene's O



Algebraic Data Types as F-Algebras

$$
\begin{array}{l}
\text{data } \mathbb{N} : \text{Set where} \\
\quad \begin{array}{lcl} \mathsf{0} & : & \mathbb{N} \\ \mathrm{S} & : & \mathbb{N} \to \mathbb{N} \end{array}
\end{array}
$$

can be rewritten as

$$
\begin{array}{l}
\text{data } \mathbb{N} : \text{Set where} \\
\quad \begin{array}{lcl} \text{intro} & : & (1 + \mathbb{N}) \to \mathbb{N} \end{array}
\end{array}
$$

or with

$$F(X) := 1 + X$$

$$
\begin{array}{l}
\text{data } \mathbb{N} : \text{Set where} \\
\quad \begin{array}{lcl} \text{intro} & : & F(\mathbb{N}) \to \mathbb{N} \end{array}
\end{array}
$$

Algebraic Data Types as F-Algebras

$$\begin{array}{l}\text{data NatList} : \text{Set where} \\ \quad \begin{array}{lcl} \text{nil} & : & \text{NatList} \\ \text{cons} & : & (\mathbb{N} \times \text{NatList}) \to \text{NatList} \end{array}\end{array}$$

can be with

$$F(X) := 1 + (\mathbb{N} \times X)$$

rewritten as

$$\begin{array}{l}\text{data NatList} : \text{Set where} \\ \quad \begin{array}{lcl} \text{intro} & : & F(\text{NatList}) \to \text{NatList} \end{array}\end{array}$$

Algebraic Data Types as F-Algebras

Finally

$$\begin{array}{l}
\mathrm{data}\ \mathrm{KleeneO} : \mathrm{Set}\ \ \mathrm{where} \\
\quad \begin{array}{lcl}
\mathsf{0} & : & \mathrm{KleeneO} \\
\mathrm{S} & : & \mathrm{KleeneO} \to \mathrm{KleeneO} \\
\mathsf{lim} & : & (\mathbb{N} \to \mathrm{KleeneO}) \to \mathrm{KleeneO}
\end{array}
\end{array}$$

can be with

$$F(X) := 1 + X + (\mathbb{N} \to X)$$

rewritten as

$$\begin{array}{l}
\mathrm{data}\ \mathrm{KleeneO} : \mathrm{Set}\ \ \mathrm{where} \\
\quad \begin{array}{lcl}
\mathrm{intro} & : & F(\mathrm{KleeneO}) \to \mathrm{KleeneO}
\end{array}
\end{array}$$

Initial F-Algebras

Initial F-Algebras F^* are minimal F-Algebras:

$$\begin{tikzcd}
\mathrm{F}(\mathrm{F}^*) \arrow[r, "\mathrm{intro}"] \arrow[d, "\mathrm{F}(g)"'] & \mathrm{F}^* \arrow[d, "\exists!\ g"] \\
\mathrm{F}(A) \arrow[r, "f"] & A
\end{tikzcd}$$

Iteration

Existence of g corresponds to iteration (example $\mathbb{N}$):

$$\begin{tikzcd} 1+\mathbb{N} \arrow[r, "\mathrm{intro}"] \arrow[d, "1+g"'] & \mathbb{N} \arrow[d, "\exists g"] \\ 1+A \arrow[r, "f"] & A \end{tikzcd}$$

$$\begin{array}{lcl} g(0) & = & g(\mathrm{intro}(\mathrm{inl})) \\ & = & f(\mathrm{inl}) \\ \\ g(\mathrm{S}(n)) & = & g(\mathrm{intro}(\mathrm{inr}(n))) \\ & = & f(\mathrm{inr}(g(n))) \end{array}$$

So with $a_0 := f(\mathrm{inl})$ and $f_0(x) := f(\mathrm{inr}(x))$

$$g(n) = f_0^n(a_0)$$

Recursion

The principle of recursion can be derived using uniqueness:
Assume

$$\begin{array}{lcl} a_0 & : & A \\ f_0 & : & (\mathbb{N} \times A) \to A \end{array}$$

We derive $g : \mathbb{N} \to A$ s.t.

$$\begin{array}{lcl} g(0) & = & a_0 \\ g(\mathrm{S}(n)) & = & f_0(n, g(n)) \end{array}$$

This allows to define e.g.

$$\begin{array}{lcl} \mathrm{pred} & : & \mathbb{N} \to \mathbb{N} \\ \mathrm{pred}(0) & = & 0 \\ \mathrm{pred}(\mathrm{S}(n)) & = & n \end{array}$$

Recursion

$$
\begin{array}{lcl}
a_0 & : & A \\
f_0 & : & (\mathbb{N} \times A) \to A
\end{array}
$$

Define $f : (1 + (\mathbb{N} \times A)) \to (\mathbb{N} \times A)$

$$
\begin{array}{lcl}
f(\mathrm{inl}) & = & (0, a_0) \\
f(\mathrm{inr}(n, a)) & = & (\mathrm{S}(n), f_0(n, a))
\end{array}
$$

$$
\begin{tikzcd}
1+\mathbb{N} \arrow[r, "\mathrm{intro}"] \arrow[d, "1+g"'] & \mathbb{N} \arrow[d, "g"] \\
1+(\mathbb{N}\times A) \arrow[r, "f"] \arrow[d, "1+\pi_0"'] & \mathbb{N}\times A \arrow[d, "\pi_0"] \\
1+\mathbb{N} \arrow[r, "\mathrm{intro}"] & \mathbb{N}
\end{tikzcd}
$$

Both $\pi_0 \circ g$ and id make the outermost diagram commute.
By uniqueness follows $\pi_0 \circ g = \mathrm{id}$,
therefore $g(n) = (n, g_0(n))$ for some $g_0 : \mathbb{N} \to A$.
Therefore

$$
\begin{aligned}
g_0(0) &= \pi_1(g(\mathrm{intro}(\mathrm{inl}))) &&= \pi_1(f(\mathrm{inl})) &&= a_0 \\
g_0(\mathrm{S}(n)) &= \pi_1(g(\mathrm{intro}(\mathrm{inr}(n)))) &&= \pi_1(f(\mathrm{inr}(n, g_0(n)))) &&= f_0(n, g_0(n))
\end{aligned}
$$

Induction

Induction can be regarded as dependent elimination:
Assume

$$\begin{array}{lcl} A & : & \mathbb{N} \to \mathrm{Set} \\ a_0 & : & A(0) \\ f_0 & : & (n : \mathbb{N}) \to A(n) \to A(\mathrm{S}(n)) \end{array}$$

We derive $g : (n : \mathbb{N}) \to A(n)$ s.t.

$$\begin{array}{lcl} g(0) & = & a_0 \\ g(\mathrm{S}(n)) & = & f_0(n, g(n)) \end{array}$$

Can be derived in the same way as recursion.

Coalgebras

Final coalgebras F^∞ are obtained by reversing the arrows in the diagram for F-algebras:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\exists! g"'] & F(A) \arrow[d, "F(g)"] \\ F^\infty \arrow[r, "\mathrm{case}"] & F(F^\infty) \end{tikzcd}$$

Coalgebras

Consider Streams $= F^\infty$ where $F(X) = \mathbb{N} \times X$:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\exists ! g"'] & \mathbb{N} \times A \arrow[d, "\mathrm{id} \times g"] \\
\mathrm{Stream} \arrow[r, "\mathrm{case}"] & \mathbb{N} \times \mathrm{Stream}
\end{tikzcd}
$$

Let

$$\mathrm{case}(s) \quad = \quad (\mathrm{head}(s), \mathrm{tail}(s))$$

and

$$f(a) = (f_0(a), f_1(a))$$

Guarded Recursion

$$\begin{tikzcd} A \arrow[r, "(f_0, f_1)"] \arrow[d, "\exists! g"'] & \mathbb{N} \times A \arrow[d, "\mathrm{id} \times g"] \\ \mathrm{Stream} \arrow[r, "(\mathrm{head}, \mathrm{tail})"] & \mathbb{N} \times \mathrm{Stream} \end{tikzcd}$$

Resulting equations:

$$\begin{array}{lcl} \mathrm{head}(g(a)) & = & \pi_0(\mathrm{case}(g(a))) = \pi_0(f_0(a), g(f_1(a))) = f_0(a) \\ \mathrm{tail}(g(a)) & = & \pi_1(\mathrm{case}(g(a))) = \pi_1(f_0(a), g(f_1(a))) = g(f_1(a)) \end{array}$$

Example of Guarded Recursion

$$\begin{array}{lcl} \mathrm{head}(g(a)) & = & f_0(a) \\ \mathrm{tail}(g(a)) & = & g(f_1(a)) \end{array}$$

describes a schema of guarded recursion (or better coiteration)
As an example, with $A = \mathbb{N}$, $f_0(n) = n$, $f_1(n) = n + 1$ we obtain:

$$\begin{array}{lcl} \multicolumn{3}{l}{\mathrm{inc} : \mathbb{N} \to \mathrm{Stream}} \\ \mathrm{head}(\mathrm{inc}(n)) & = & n \\ \mathrm{tail}(\ \mathrm{inc}(n)) & = & \mathrm{inc}(n + 1) \end{array}$$

Corecursion

Corecursion allows for tail to escape into a stream.
Example:

$$\begin{array}{rcl} \multicolumn{3}{l}{\mathrm{cons} : (\mathbb{N} \times \mathrm{Stream}) \to \mathrm{Stream}} \\ \mathrm{head}(\mathrm{cons}(n, s)) & = & n \\ \mathrm{tail}(\ \mathrm{cons}(n, s)) & = & s \end{array}$$

Corecursion

More generally, if

$$\begin{array}{lll} A & : & \mathrm{Set} \\ f_0 & : & A \to \mathbb{N} \\ f_1 & : & A \to (\mathrm{Stream} + A) \end{array}$$

we get $g : A \to \mathrm{Stream}$ s.t.

$$\begin{array}{lllll} \mathrm{head}(g(a)) & = & f_0(a) & & \\ \mathrm{tail}(\ g(a)) & = & s & \text{if} & f_1(a) = \mathrm{inl}(s) \\ \mathrm{tail}(\ g(a)) & = & g(a') & \text{if} & f_1(a) = \mathrm{inr}(a') \end{array}$$

Corecursion vs Recursion

Compare with recursion which allowed for

$$
\begin{array}{lcl}
A & : & \mathrm{Set} \\
a_0 & : & A \\
f_0 & : & (\mathbb{N} \times A) \to A
\end{array}
$$

To define

$$
\begin{array}{lcl}
g & : & \mathbb{N} \to A \\
g(0) & = & a_0 \\
g(\mathrm{S}(n)) & = & f_0(n, g(n))
\end{array}
$$

Nested Corecursion

$$
\begin{array}{lcl}
\mathrm{stutter} : \mathbb{N} \to \mathrm{Stream} & & \\
\mathrm{head}(\qquad \mathrm{stutter}(n)) & = & n \\
\mathrm{head}(\mathrm{tail}(\mathrm{stutter}(n))) & = & n \\
\mathrm{tail}(\ \ \mathrm{tail}(\mathrm{stutter}(n))) & = & \mathrm{stutter}(n+1)
\end{array}
$$

Even more general schemas can be defined.

Weakly Final Coalgebra

- Equality for final coalgebras is undecidable:
  Two streams

$$\begin{array}{lllllllll} s & = & (a_0 & , & a_1 & , & a_2 & , & \ldots \\ t & = & (b_0 & , & b_1 & , & b_2 & , & \ldots \end{array}$$

  are equal iff $a_i = b_i$ for all i.
- Even the weak assumption

$$\forall s.\exists n, s'.s = \mathrm{cons}(n, s')$$

  results in an undecidable equality.
- Weakly final coalgebras obtained by omitting uniqueness of g in diagram for coalgebras.
- However, one can extend schema of coiteration as above, and still preserve decidability of equality.
  - Those schemata are usually not derivable in weakly final coalgebras.

Patterns and Copatterns

- We can define now functions by patterns and copatterns.
- Example define stream:

$$f(n) = n, n, n{-}1, n{-}1, \ldots 0, 0, N, N, N{-}1, N{-}1, \ldots 0, 0, N, N, N{-}1, N{-}$$

- Step 1:

$$\begin{array}{lcl} f & : & \mathbb{N} \to \mathrm{Stream} \\ f & = & ? \end{array}$$

- Step 2: Apply f to variable:

$$f(n) \quad = \quad ?$$

- Step 3: Make **copattern matching** on $f(n) : \mathrm{Stream}$:

$$\begin{array}{lcl} \mathrm{head}(f(n)) & = & ? \\ \mathrm{tail}(\ \ f(n)) & = & ? \end{array}$$

Patterns and Copatterns

- We make **pattern matching** on $n = 0$, $n = \mathrm{S}(m)$:

$$\begin{array}{lcl} \mathrm{head}(f(0)) & = & 0 \\ \mathrm{tail}(\ \ f(0)) & = & ? \\ \mathrm{head}(f(\mathrm{S}(n))) & = & \mathrm{S}(n) \\ \mathrm{tail}(\ \ f(\mathrm{S}(n))) & = & ? \end{array}$$

Patterns and Copatterns

$$f(n) = n, n, n{-}1, n{-}1, \ldots 0, 0, N, N, N{-}1, N{-}1, \ldots 0, 0, N, N, N{-}1, N{-}1, \ldots$$

- We make **copattern matching** on Stream:

$$\begin{array}{lcl}
\mathrm{head}(f(0)) & = & 0 \\
\mathrm{head}(\mathrm{tail}(f(0))) & = & 0 \\
\mathrm{tail}(\mathrm{tail}(f(0))) & = & f(N) \\
\mathrm{head}(f(\mathrm{S}(n))) & = & \mathrm{S}(n) \\
\mathrm{head}(\mathrm{tail}(f(\mathrm{S}(n)))) & = & \mathrm{S}(n) \\
\mathrm{tail}(\mathrm{tail}(f(\mathrm{S}(n)))) & = & f(n)
\end{array}$$

Results of paper in POPL

- Development of a recursive simply typed calculus (no termination check).
- Allows to derive schemas for pattern/copattern matching.
- Proof that subject reduction holds.

$$t : A, \quad t \longrightarrow t' \text{ implies } t' : A$$

Codata Type

- Idea of Codata Types:

$$\begin{array}{l} \mathrm{codata\ Stream : Setwhere} \\ \quad \mathrm{cons} \quad : \quad \mathbb{N} \times \mathrm{Stream} \to \mathrm{Stream} \end{array}$$

- Theoretical problem:
  Underlying assumption is

  $$\forall s : \mathrm{Stream}.\exists n, s'.s = \mathrm{cons}(n, s')$$

  which results in undecidable equality.
- Results in Coq in a long known problem of subject reduction.
- In Agda severe restriction of elimination for coalgebras, which makes proving formulas involving coalgebras very difficult.

Problem of Subject reduction

$\text{data } \_{==}\_ \ \{A : \text{Set}\}\ (a : A) : A \to \text{Set where}$
$\quad \text{refl} : a == a$

$\text{codata Stream} : \text{Set where}$
$\quad \text{cons} : (\mathbb{N} \times \text{Stream}) \to \text{Stream}$

$\mathsf{zeros} : \text{Stream}$
$\mathsf{zeros} = \text{cons}(0, \mathsf{zeros})$

$\mathsf{force} : \text{Stream} \to \text{Stream}$
$\mathsf{force}(s) = \text{case } s \text{ of cons}(x, y) \to \text{cons}(x, y)$

$\mathsf{lem1} : (s : \text{Stream}) \to s == \mathsf{force}(s))$
$\mathsf{lem1}(s) = \text{case } s \text{ of cons}(x, y) \to \text{refl}$

$\mathsf{lem2} : \mathsf{zeros} == \text{cons}(0, \mathsf{zeros})$
$\mathsf{lem2} = \mathsf{lem1}(\mathsf{zeros}) \longrightarrow \text{refl} \quad \neg(\text{refl} : \mathsf{zeros} == \text{cons}(0, \mathsf{zeros}))$

Multiple Constructors in Algebras and Coalgebras

- Several constructors in algebras correspond to disjoint union:

$$\begin{array}{l} \mathrm{data}\ \mathbb{N} : \mathrm{Set}\ \mathrm{where} \\ \quad \begin{array}{lcl} \mathsf{0} & : & \mathbb{N} \\ \mathrm{S} & : & \mathbb{N} \to \mathbb{N} \end{array} \end{array}$$

  corresponds to

$$\begin{array}{l} \mathrm{data}\ \mathbb{N} : \mathrm{Set}\ \mathrm{where} \\ \quad \begin{array}{lcl} \mathrm{intro} & : & (1 + \mathbb{N}) \to \mathbb{N} \end{array} \end{array}$$

Multiple Constructors in Algebras and Coalgebras

- Dual of disjoint union is products, and therefore multiple destructors correspond to product:

$$
\begin{array}{l}
\mathrm{coalg}\ \mathrm{Stream} : \mathrm{Set}\ \mathrm{where} \\
\quad \begin{array}{lcl}
\mathrm{head} & : & \mathrm{Stream} \to \mathbb{N} \\
\mathrm{tail} & : & \mathrm{Stream} \to \mathrm{Stream}
\end{array}
\end{array}
$$

corresponds to

$$
\begin{array}{l}
\mathrm{coalg}\ \mathrm{Stream} : \mathrm{Set}\ \mathrm{where} \\
\quad \begin{array}{lcl}
\mathrm{case} & : & \mathrm{Stream} \to (\mathbb{N} \times \mathrm{Stream})
\end{array}
\end{array}
$$

Codata Types Correspond to Disjoint Union

- Consider

$$\begin{array}{l} \mathrm{codata}\ \mathrm{coList} : \mathrm{Set}\ \mathrm{where} \\ \quad \begin{array}{lcl} \mathrm{nil} & : & \mathrm{coList} \\ \mathrm{cons} & : & (\mathbb{N} \times \mathrm{coList}) \to \mathrm{coList} \end{array} \end{array}$$

- Cannot be simulated by a coalgebra with several destructors.

Simulating Codata Types by Simultaneous Algebras/Coalgebras

- Represent Codata as follows

mutual

$$
\begin{array}{l}
\quad \mathrm{coalg}\ \mathrm{coList} : \mathrm{Set}\ \mathrm{where} \\
\quad\quad \mathrm{unfold} : \mathrm{coList} \to \mathrm{coListShape} \\
\\
\quad \mathrm{data}\ \mathrm{coListShape} : \mathrm{Set}\ \mathrm{where} \\
\quad\quad \begin{array}{lcl}
\mathrm{nil} & : & \mathrm{coListShape} \\
\mathrm{cons} & : & (\mathbb{N} \times \mathrm{coList}) \to \mathrm{coListShape}
\end{array}
\end{array}
$$

Example Append

$$\begin{array}{l}\mathrm{append} : (\mathrm{coList} \times \mathrm{coList}) \to \mathrm{coList} \\ \mathrm{append}(l, l') = \mathrm{appendaux}(\mathrm{unfold}(l), l')\end{array}$$

$$\begin{array}{lcl}\multicolumn{3}{l}{\mathrm{appendaux} : (\mathrm{coListShape} \times \mathrm{coList}) \to \mathrm{coList}} \\ \mathrm{appendaux}(\mathrm{nil}, l') & = & l' \\ \mathrm{unfold}(\mathrm{appendaux}(\mathrm{cons}(n, l), l')) & = & \mathrm{cons}(n, \mathrm{append}(l, l'))\end{array}$$

Conclusion

- Symmetry between
  - algebras and coalgebras,
  - iteration and coiteration,
  - recursion and corecursion,
  - patterns and copatterns.
- Unknown: dual of induction (requires codependent types?)
- Codata construct assumes every element is introduced by a constructor, which results in undecidable equality.
- Weakly final coalgebras solves this problem, but adds small overhead when programming.