Pattern and Copattern matching

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Iteration, Recursion, Induction

Coiteration, Corecursion

Bisimilarity and Coinduction

Proofs by Coinduction of Bisimilarity in Transition Systems

Mixed Patterns and Copatterns

Unnesting of Pattern/Copattern Matching

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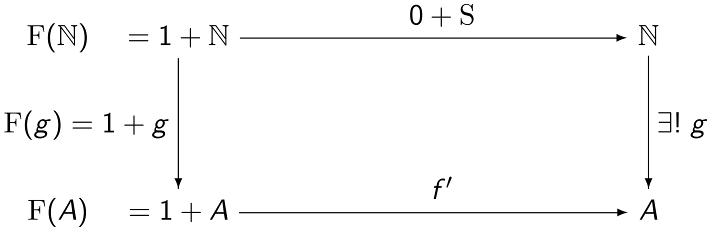
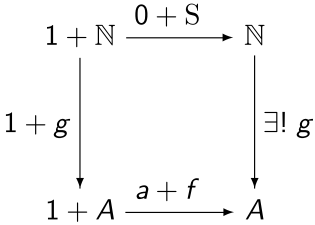
Unnesting of Pattern/Copattern Matching

$\mathbb{N}$ as an Initial Algebra

- $\mathbb{N}$ is initial algebra of the functor $\mathrm{F}(X) = 1 + X$
- 

  f' can be decomposed as $f' = a + f$

Unique Iteration



Unique existence of g means **unique iteration**:

Given $a : A$ and $f : A \to A$ there exists a unique

$$
\begin{array}{lcl}
g : \mathbb{N} \to A & & \\
g\ 0 & = & a \\
g\ (\mathrm{S}\ n) & = & f\ (g\ n) \\
\text{i.e} & & \\
g\ (\mathrm{S}^n\ 0) & = & f^n\ a
\end{array}
$$

Unique Recursion

- From the principle of unique iteration we can prove the principle of unique (primitive) recursion:

  Given $a : A$ and $f : \mathbb{N} \to A \to A$ there exists a unique

$$
\begin{array}{lcl}
g : \mathbb{N} \to A & & \\
g\ 0 & = & a \\
g\ (\mathrm{S}\ n) & = & f\ n\ (g\ n)
\end{array}
$$

Induction

- From the principle of unique iteration we can prove the principle of induction:

  Assume $A : \mathbb{N} \to \mathrm{Set}$, $a : A\ 0$ and $f : (n : \mathbb{N}) \to A\ n \to A\ (\mathrm{S}\ n)$

  There exists a unique

$$
\begin{array}{lcl}
\multicolumn{3}{l}{g : (n : \mathbb{N}) \to A\ n} \\
g\ 0 & = & a \\
g\ (\mathrm{S}\ n) & = & f\ n\ (g\ n)
\end{array}
$$

- Using induction we can prove that if we have two solutions for a iteration or recursion principle, they are pointwise equal, i.e. uniqueness of iteration and recursion.

Pattern Matching

- The above means that we can define

$$
\begin{array}{lcll}
g : (n : \mathbb{N}) \to A\ n & & & \\
g\ 0 & = & a & \text{for some } a : A \\
g\ (\mathrm{S}\ n) & = & a' & \text{for some } a' : A \text{ depending on } n
\end{array}
$$

  where in the second case we can use the **recursion hypothesis** or **induction hypothesis** $g\ n$.
- This means we can define $g\ n$ by **pattern matching** on $n : \mathbb{N}$.

Iteration, Recursion, Induction

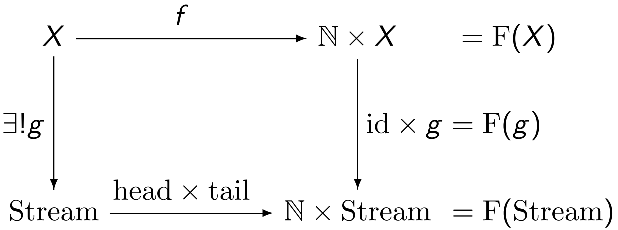
Theorem

Assume $\mathbb{N} : \mathrm{Set}$, $0 : \mathbb{N}$, $\mathrm{S} : \mathbb{N} \to \mathbb{N}$.
The following are equivalent

- *The principle of unique iteration.*
- *The principle of unique recursion.*
- *The principle of unique induction.*
- *The principle of induction.*

Streams as a Final Coalgebra

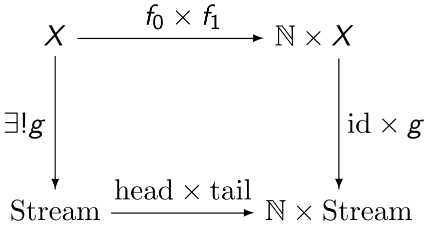
- Dual of $+$ is $\times$, so we use for clarity a functor using product rather than disjoint union:
- Stream is the final coalgebra of $\mathrm{F}(X) = \mathbb{N} \times X$



- We can decompose f as

$$f \quad = \quad f_0 \times f_1$$

Unique Coiteration



This corresponds to the principle of unique coiteration:
There exists a unique

$$\begin{array}{lcl} g : A \to \mathrm{Stream} & & \\ \mathrm{head}\ (g\ x) & = & f_0\ x \\ \mathrm{tail}\ \ (g\ x) & = & g\ (f_1\ x) \end{array}$$

Unique Coiteration

- We had:

$$\begin{array}{lcl} \mathrm{head}\ (g\ x)) & = & f_0\ x \\ \mathrm{tail}\ \ (g\ x) & = & g\ (f_1\ x) \end{array}$$

- By choosing f_0, f_1 we can define $g : X \to \mathrm{Stream}$ s.t.

$$\begin{array}{lllll} \mathrm{head} & (g\ x) & = & n & \text{for some } n : \mathbb{N} \text{ depending on } x \\ \mathrm{tail} & (g\ x) & = & g\ x' & \text{for some } x' : X \text{ depending on } x \end{array}$$

Unique Corecursion

- From unique coiteration we can derive **unique corecursion**:
  There exists a unique

$$
\begin{array}{lllll}
g : A \to \mathrm{Stream} & & & & \\
\mathrm{head} & (g\ x) & = & n & \text{for some } n : \mathbb{N} \text{ depending on } x \\
\mathrm{tail} & (g\ x) & = & g\ x' & \text{for some } x' : X \text{ depending on } x \\
 & & & \text{or} & \\
 & & = & s & \text{for some } s : \mathrm{Stream} \text{ depending on } x
\end{array}
$$

- This means we can define $g\ x$ by **copattern matching**

Examples

- We can define

$$
\begin{array}{llcl}
\mathrm{cons} & & : & (\mathbb{N} \times \mathrm{Stream}) \to \mathrm{Stream} \\
\mathrm{head} & (\mathrm{cons}(n, s)) & = & n \\
\mathrm{tail} & (\mathrm{cons}(n, s)) & = & s
\end{array}
$$

Note: cons not primitive but **defined** by corecursion

$$
\begin{array}{llcl}
\mathrm{inc} & & : & \mathbb{N} \to \mathrm{Stream} \\
\mathrm{head} & (\mathrm{inc}\ n) & = & n \\
\mathrm{tail} & (\mathrm{inc}\ n) & = & \mathrm{inc}\ (n+1)
\end{array}
$$

Examples

$$
\begin{array}{llcl}
\mathrm{inc}' & & : & \mathbb{N} \to \mathrm{Stream} \\
\mathrm{head} & (\mathrm{inc}'(n)) & = & n \\
\mathrm{tail} & (\mathrm{inc}'(n)) & = & \mathrm{inc}''(n+1)
\end{array}
$$

$$
\begin{array}{llcl}
\mathrm{inc}'' & & : & \mathbb{N} \to \mathrm{Stream} \\
\mathrm{head} & (\mathrm{inc}''(n)) & = & n \\
\mathrm{tail} & (\mathrm{inc}''(n)) & = & \mathrm{inc}'(n+1)
\end{array}
$$

Bisimilarity

- Bisimilarity $\sim$ on Streams is an **indexed final coalgebra**.
- Consider the category $\mathrm{Set}^{\mathrm{Stream} \times \mathrm{Stream}}$ of binary relations

$$\varphi : \mathrm{Stream} \times \mathrm{Stream} \to \mathrm{Set}$$

- Let

$$\begin{array}{l} \mathrm{F}^{\sim} \ : \ \mathrm{Set}^{\mathrm{Stream} \times \mathrm{Stream}} \to \mathrm{Set}^{\mathrm{Stream} \times \mathrm{Stream}} \\ \mathrm{F}^{\sim}(\varphi, (s, s')) = (\mathrm{head}\ s = \mathrm{head}\ s') \times \varphi\ (\mathrm{tail}\ s, \mathrm{tail}\ s') \end{array}$$

Bisimilarity

- That $\sim$ is a $\mathrm{F}^{\sim}$ coalgebra means there exist

$$\begin{aligned}\mathrm{elim}_{\sim} : & (s, s' : \mathrm{Stream}) \\ & \to s \sim s' \\ & \to (\mathrm{head}\ s = \mathrm{head}\ s') \times (\mathrm{tail}\ s \sim \mathrm{tail}\ s')\end{aligned}$$

  i.e.

$$s \sim s' \to (\mathrm{head}\ s = \mathrm{head}\ s') \wedge ((\mathrm{tail}\ s) \sim (\mathrm{tail}\ s'))$$

- Let $\mathrm{elim}^0_{\sim}$ and $\mathrm{elim}^1_{\sim}$ the two components of $\mathrm{elim}_{\sim}$,

$$\begin{aligned}\mathrm{elim}^0_{\sim} &: (s, s' : \mathrm{Stream}) \to s \sim s' \to \mathrm{head}\ s = \mathrm{head}\ s' \\ \mathrm{elim}^1_{\sim} &: (s, s' : \mathrm{Stream}) \to s \sim s' \to \mathrm{tail}\ s \sim \mathrm{tail}\ s'\end{aligned}$$

  and hide the first two arguments of $\mathrm{elim}^i_{\sim}$.

Bisimilarity

- That $\sim$ is a final $\mathrm{F}^{\sim}$-coalgebra means that it is the largest such relation:

$$
\begin{array}{ccc}
\varphi\,(s,s') & \xrightarrow{\quad f \quad} & \mathrm{head}\ s = \mathrm{head}\ s' \wedge \varphi\,(\mathrm{tail}\ s, \mathrm{tail}\ s') \\
\Big\downarrow{\scriptstyle \exists! g} & & \Big\downarrow{\scriptstyle \mathrm{id} \wedge g} \\
s \sim s' & \xrightarrow{\ \mathrm{elim}_{\sim}\ } & \mathrm{head}\ s = \mathrm{head}\ s' \wedge (\mathrm{tail}\ s) \sim (\mathrm{tail}\ s')
\end{array}
$$

- This means that

$$\forall s, s'.\varphi\,(s,s') \rightarrow \mathrm{head}\ s = \mathrm{head}\ s' \wedge \varphi\,(\mathrm{tail}\ s, \mathrm{tail}\ s')$$

then

$$\forall s, s'.\varphi\,(s,s') \rightarrow s \sim s'$$

Bisimilarity

- So we have

$$s \sim s' \to \text{head } s = \text{head } s' \wedge (\text{tail } s) \sim (\text{tail } s')$$

and if

$$\forall s, s'.\varphi\,(s, s') \to \text{head } s = \text{head } s' \wedge \varphi\,(\text{tail } s, \text{tail } s')$$

then

$$\forall s, s'.\varphi\,(s, s') \to s \sim s'$$

Corecursive Proof of Bisimilarity

- Because $\sim$ is a final coalgebra we can compute proofs of it by corecursion:
- We can define

$$
\begin{array}{lcl}
f : (s, s' : \mathrm{Stream}) \to \varphi\ s\ s' \to s \sim s' \\
\mathrm{elim}^0_\sim\ (f\ s\ s'\ x) & = & \text{an element of } \mathrm{head}\ s = \mathrm{head}\ s' \\
\mathrm{elim}^0_\sim\ (f\ s\ s'\ x) & = & \text{an element of } (\mathrm{tail}\ s) \sim (\mathrm{tail}\ s')
\end{array}
$$

where in the last line we can use

- either a proof of $\mathrm{tail}\ s \sim \mathrm{tail}\ s'$ defined before
- or use the corecursion hypothesis $f\ (\mathrm{tail}\ s)\ (\mathrm{tail}\ s')\ x'$ for some $x' : \varphi\ (\mathrm{tail}\ s)\ (\mathrm{tail}\ s')$

Coinduction

Theorem

Assume $\mathrm{Stream} : \mathrm{Set}$, $\mathrm{head} : \mathrm{Stream} \to \mathbb{N}$, $\mathrm{tail} : \mathrm{Stream} \to \mathrm{Stream}$.
The following are equivalent

- *The principle of unique coiteration.*
- *The principle of unique corecursion.*
- *The principle of iteration together with the principle that bisimilarity $\sim$ implies equality*

$$\forall s, s' : \mathrm{Stream}. s \sim s' \to s = s'$$

Because of the possibility of defining elements of $s \sim s'$ the latter can be considered as a **principle of coinduction**.

Principle of Coinduction

- Let $\varphi : \mathrm{Stream} \to \mathrm{Stream} \to \mathrm{Set}$.
- We can prove

$$\forall s, s' : \mathrm{Stream}.\varphi\ s\ s' \to s = s'$$

by showing

$$\begin{array}{l}\forall s, s' : \mathrm{Stream}.\varphi\ s\ s' \to \mathrm{head}\ s = \mathrm{head}\ s' \\ \forall s, s' : \mathrm{Stream}.\varphi\ s\ s' \to \mathrm{tail}\ s = \mathrm{tail}\ s'\end{array}$$

where for proving $\mathrm{tail}\ s = \mathrm{tail}\ s'$ we can use the coinduction hypothesis that $\varphi\ (\mathrm{tail}\ s)\ (\mathrm{tail}\ s')$ implies $\mathrm{tail}\ s = \mathrm{tail}\ s'$.

Indexed Coinduction

- Instead of defining φ as a predicate $\mathrm{Stream} \to \mathrm{Stream} \to \mathrm{Set}$ we can assume

  $A : \mathrm{Set}$
  $s, t : A \to \mathrm{Stream}$
  and define
  $\varphi\ s'\ t' = (a : A) \times (s' = s\ a) \times (t' = t\ a)$

- Coinduction of φ becomes then the principle of indexed coinduction (see next slide)

Indexed Coinduction

- Assume

$$\begin{array}{lcl} A & : & \mathrm{Set} \\ s_0, s_1 & : & A \to \mathrm{Stream} \end{array}$$

- We can prove

$$\forall a : A. s_0\ a = s_1\ a$$

by showing

$$\begin{array}{l} \forall a : A. \mathrm{head}\ (s\ a) = \mathrm{head}\ (t\ a) \\ \forall a : A. \mathrm{tail}\ (s\ a) = \mathrm{tail}\ (t\ a) \end{array}$$

where for proving $\mathrm{tail}\ (s\ a) = \mathrm{tail}\ (t\ a)$ we can use that $\mathrm{tail}\ (s\ a) = s\ a'$ and $\mathrm{tail}\ (t\ a) = t\ a'$ and therefore by **coinduction-hypothesis** $s\ a' = t\ a'$.

Example Proof by Coinduction

- Remember

$$
\begin{array}{lcl}
\mathrm{inc} & : & \mathbb{N} \to \mathrm{Stream} \\
\mathrm{head}(\mathrm{inc}\ n) & = & n \\
\mathrm{tail}\ \ (\mathrm{inc}\ n) & = & \mathrm{inc}\ (n+1)
\end{array}
$$

$$
\begin{array}{lcl}
\mathrm{inc}' & : & \mathbb{N} \to \mathrm{Stream} \\
\mathrm{head}(\mathrm{inc}'(n)) & = & n \\
\mathrm{tail}\ \ (\mathrm{inc}'(n)) & = & \mathrm{inc}''(n+1)
\end{array}
$$

$$
\begin{array}{lcl}
\mathrm{inc}'' & : & \mathbb{N} \to \mathrm{Stream} \\
\mathrm{head}(\mathrm{inc}''(n)) & = & n \\
\mathrm{tail}\ \ (\mathrm{inc}''(n)) & = & \mathrm{inc}'(n+1)
\end{array}
$$

Example Proof by Coinduction

- We show
$$\forall n \in \mathbb{N}.\mathrm{inc}'\ n = \mathrm{inc}\ n \wedge \mathrm{inc}''\ n = \mathrm{inc}\ n$$
- Formally we would use in the above

$$\begin{array}{l} A = \mathbb{N} + \mathbb{N} \\ s\ (\mathrm{inl}\ n) = \quad \mathrm{inc}'\ n \\ s\ (\mathrm{inr}\ n) = \quad \mathrm{inc}''\ n \\ t\ (\mathrm{inl}\ n) = \quad \mathrm{inc}\ n \\ t\ (\mathrm{inr}\ n) = \quad \mathrm{inc}\ n \end{array}$$

and show

$$\forall a : A. s\ a = t\ a$$

Example Proof by Coinduction

- Proof of

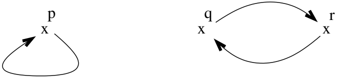
$$\forall n \in \mathbb{N}.\mathrm{inc}'\ n = \mathrm{inc}\ n \wedge \mathrm{inc}''\ n = \mathrm{inc}\ n$$

- Assume $n : \mathbb{N}$.

$$\begin{array}{l}
\mathrm{head}\ (\mathrm{inc}'\ n) = n = \mathrm{head}\ (\mathrm{inc}\ n) \\
\mathrm{head}\ (\mathrm{inc}''\ n) = n = \mathrm{head}\ (\mathrm{inc}\ n) \\
\mathrm{tail}\ (\mathrm{inc}'\ n) = \mathrm{inc}''\ (n+1) \stackrel{\mathrm{co-IH}}{=} \mathrm{inc}\ (n+1) = \mathrm{tail}\ (\mathrm{inc}\ n) \\
\mathrm{tail}\ (\mathrm{inc}''\ n) = \mathrm{inc}'\ (n+1) \stackrel{\mathrm{co-IH}}{=} \mathrm{inc}\ (n+1) = \mathrm{tail}\ (\mathrm{inc}\ n)
\end{array}$$

Bisimilarity

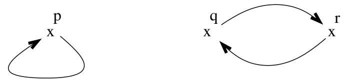
- Consider the following (unlabelled) transition system:



- Bisimilarity is the final coalgebra

$$
\begin{aligned}
p \sim q \rightarrow (\forall p'.p \longrightarrow p' \\
\rightarrow \exists q'.q \longrightarrow q' \wedge p' \sim q') \\
\wedge \cdots \text{symmetric case} \cdots \}
\end{aligned}
$$

Proof using the Definition of $\sim$



- We show $p \sim q \land p \sim r$ by coinduction:
- **Coinduction step for** $p \sim q$:
  - Assume $p \longrightarrow p'$. Then $p' = p$.
    We have $q \longrightarrow r$ and by co-IH $p \sim r$.
  - Assume $q \longrightarrow q'$. Then $q' = r$.
    We have $p \longrightarrow p$ and by co-IH $p \sim r$.
- **Coinduction step for** $p \sim r$:
  - Assume $p \longrightarrow p'$. Then $p' = p$.
    We have $r \longrightarrow q$ and by co-IH $p \sim q$.
  - Assume $r \longrightarrow r'$. Then $r' = q$.
    We have $p \longrightarrow p$ and by co-IH $p \sim q$.

Traditional Argument of Proving Bisimiliarity

- The standard argument for showing $p \sim q \land p \sim r$ is as follows:
  Define a relation φ on states by

  $$\varphi(p', q') \Leftrightarrow p' = p \land (q' = q \lor q' = r)$$

  Show φ is a simulation:

  $$\forall p, p', q.\varphi(p, q) \land p \longrightarrow p' \Rightarrow \exists q'.q \longrightarrow q' \land \varphi(p', q')$$
  $$\forall p, q, q'.\varphi(p, q) \land q \longrightarrow q' \Rightarrow \exists p'.p \longrightarrow p' \land \varphi(p', q')$$

Comparison with Proofs by Induction

- We can compare both proofs to proofs by induction on natural number. Consider a proof of

$$\forall n, m, k . n + (m + k) = (n + m) + k$$

- The traditional proof would corresponds to defining a relation

$$R(k) \Leftrightarrow \forall n, m . n + (m + k) = (n + m) + k$$

and showing

$$R(0) \wedge \forall n . R(n) \rightarrow R(\mathrm{S}(n))$$

- Although this argument and the standard inductive proof using the induction hypothesis are equivalent, the standard inductive proof is more convenient and easier to follow.
- We hope that proofs by coinduction will similarly be easier if we do it by referring to the coinduction hypothesis.

Nested Pattern Matching

- Course of Value primitive recursion allows deep pattern matching. E.g. we can define the Fibonaccie numbers

$$
\begin{array}{lcl}
\mathrm{fib} : \mathbb{N} \to \mathbb{N} & & \\
\mathrm{fib}\ 0 & = & 1 \\
\mathrm{fib}\ (\mathrm{S}\ 0) & = & 1 \\
\mathrm{fib}\ (\mathrm{S}\ (\mathrm{S}\ n)) & = & \mathrm{fib}\ n + \mathrm{fib}\ (\mathrm{S}\ n)
\end{array}
$$

- We can now even mix pattern and copattern matching.

Example Mixed Pattern/Copattern Matching

- We can define now functions by patterns and copatterns.
- Example define stream:
  $f\ n =$
  $n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

Example Mixed Pattern/Copattern Matching

$$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$$

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f \quad = \quad ? \end{array}$$

Example Mixed Pattern/Copattern Matching

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f \quad = \quad ? \end{array}$$

Copattern matching on $f : \mathbb{N} \to \mathrm{Stream}$:

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f\ n \quad = \quad ? \end{array}$$

Example Mixed Pattern/Copattern Matching

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f\ n \quad = \quad ? \end{array}$$

Copattern matching on $f\ n : \mathrm{Stream}$:

$$\begin{array}{lll} f : \mathbb{N} \to \mathrm{Stream} & & \\ \mathrm{head}\ (f\ n) & = & ? \\ \mathrm{tail}\ \ (f\ n) & = & ? \end{array}$$

Example Mixed Pattern/Copattern Matching

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f\ n \quad = \quad ? \end{array}$$

Solve first case, copattern match on second case:

$$\begin{array}{llll} f : \mathbb{N} \to \mathrm{Stream} & & & \\ \mathrm{head} & (f\ n) & = & n \\ \mathrm{head}\ (\mathrm{tail} & (f\ n)) & = & ? \\ \mathrm{tail} \quad (\mathrm{tail} & (f\ n)) & = & ? \end{array}$$

Example Mixed Pattern/Copattern Matching

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{l} f : \mathbb{N} \to \text{Stream} \\ f\ n \quad = \quad ? \end{array}$$

Solve second line, pattern match on n

$$\begin{array}{llll} f : \mathbb{N} \to \text{Stream} & & & \\ \text{head} & (f\ n) & = & n \\ \text{head} & (\text{tail}\ (f\ n)) & = & n \\ \text{tail} & (\text{tail}\ (f\ 0)) & = & ? \\ \text{tail} & (\text{tail}\ (f\ (\text{S}\ n))) & = & ? \end{array}$$

Example Mixed Pattern/Copattern Matching

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f\ n \quad = \quad ? \end{array}$$

Solve remaining cases

$$\begin{array}{llll} f : \mathbb{N} \to \mathrm{Stream} & & & \\ \mathrm{head} & (f\ n) & = & n \\ \mathrm{head} & (\mathrm{tail}\ (f\ n)) & = & n \\ \mathrm{tail} & (\mathrm{tail}\ (f\ 0)) & = & f\ N \\ \mathrm{tail} & (\mathrm{tail}\ (f\ (\mathrm{S}\ n))) & = & f\ n \end{array}$$

Results of paper in POPL (2013)

- Development of a recursive simply typed calculus (no termination check).
- Allows to derive schemata for pattern/copattern matching.
- Proof that subject reduction holds.

$$t : A, \quad t \longrightarrow t' \text{ implies } t' : A$$

- Subject reduction fails when using codata types in combination with the equality type (e.g. in Coq and early versions of Agda).

Consider Example from above

$$
\begin{array}{llll}
f : \mathbb{N} \to \mathrm{Stream} & & & \\
\mathrm{head} & (f\ n) & = & n \\
\mathrm{head} & (\mathrm{tail}\ (f\ n)) & = & n \\
\mathrm{tail} & (\mathrm{tail}\ (f\ 0)) & = & f\ N \\
\mathrm{tail} & (\mathrm{tail}\ (f\ (\mathrm{S}\ n))) & = & f\ n
\end{array}
$$

We show how this example can be reduced to unnested (co)pattern matching.
In a second step (not shown today) one can reduce it to primitive (co)recursion operators.

Unnesting of Nested (Co)Pattern Matching

We follow the steps in the pattern matching:
We start with

$$
\begin{array}{lcl}
f : \mathbb{N} \to \mathrm{Stream} & & \\
\mathrm{head}\ (f\ \ n) & = & n \\
\mathrm{tail}\ \ \ (f\ \ n) & = & ?
\end{array}
$$

Copattern matching on $\text{tail}\ (f\ n)$:

$$
\begin{array}{lllcl}
f : \mathbb{N} \to \text{Stream} & & & & \\
\text{head} & & (f\ n) & = & n \\
\text{head} & (\text{tail} & (f\ n)) & = & n \\
\text{tail} & (\text{tail} & (f\ n)) & = & ?
\end{array}
$$

corresponds to

$$
\begin{array}{lllcl}
 & f : \mathbb{N} \to \text{Stream} & & & \\
 & \text{head}\ (f\ n) & & = & n \\
 & \text{tail}\ \ (f\ n) & & = & g\ n \\
 & & & & \\
 & g : \mathbb{N} \to \text{Stream} & & & \\
(\text{head}\ (\text{tail}\ (f\ n)) \quad =) & \text{head}\ (g\ n) & & = & n \\
(\text{tail}\ \ (\text{tail}\ (f\ n)) \quad =) & \text{tail}\ \ (g\ n) & & = & ?
\end{array}
$$

Pattern matching on $\text{tail}\ (\text{tail}\ (f\ n))$:

$$\begin{array}{llll} f : \mathbb{N} \to \text{Stream} & & & \\ \text{head} & (f\ n) & = & n \\ \text{head}\ (\text{tail}\ (f\ n)) & & = & n \\ \text{tail} & (\text{tail}\ (f\ 0)) & = & f\ N \\ \text{tail} & (\text{tail}\ (f\ (\text{S}\ n))) & = & f\ n \end{array}$$

corresponds to

$$\begin{array}{lllll} & f : \mathbb{N} \to \text{Stream} & & & \\ & \text{head}\ (f\ n) & & = & n \\ & \text{tail} & (f\ n) & = & g\ n \end{array}$$

$$\begin{array}{lllll} & g : \mathbb{N} \to \text{Stream} & & & \\ (\text{head}\ (\text{tail}\ (f\ n)) \quad =) & \text{head}\ (g\ n) & & = & n \\ (\text{tail}\ (\text{tail}\ (f\ n)) \quad =) & \text{tail} & (g\ n) & = & k\ n \end{array}$$

$$\begin{array}{lllll} & k : \mathbb{N} \to \text{Stream} & & & \\ (\text{tail}\ (\text{tail}\ (f\ 0)) \quad =) & k & 0 & = & f\ N \\ (\text{tail}\ (\text{tail}\ (f\ (\text{S}\ n))) \quad =) & k & (\text{S}\ n) & = & f\ n \end{array}$$

Conclusion

- Principle of induction is well established and makes proofs much easier.
- In theoretical computer science coinductive principles occur frequently.
  - Main reason: interactive programs running continuously in various frameworks (imperative, object-oriented, process-calculi)
- Coalgebras as being defined by their eliminators rather than infinite applications of constructors makes clear when recursive calls are allowed.
- Proofs by coinduction in the above situation can be carried out similarly as proofs by induction.
- Main difficulty: when are we allowed to apply co-IH?
  - In the corecursion step we have a proof obligation, and can use the co-IH to prove it.

Conclusion

- Copattern matching as the dual of pattern matching.
  - Pattern matching is an elimination principle for inductive types (initial algebras).
  - Copattern matching is an introduction principle for coinductive types (final coalgebras).
- Mixed pattern and copattern matching can be reduced to simple pattern and copattern matching.