Coinductive Reasoning in Dependent Type Theory - Copatterns, Objects, Processes

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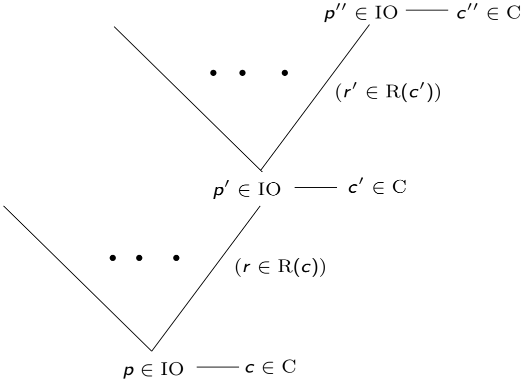
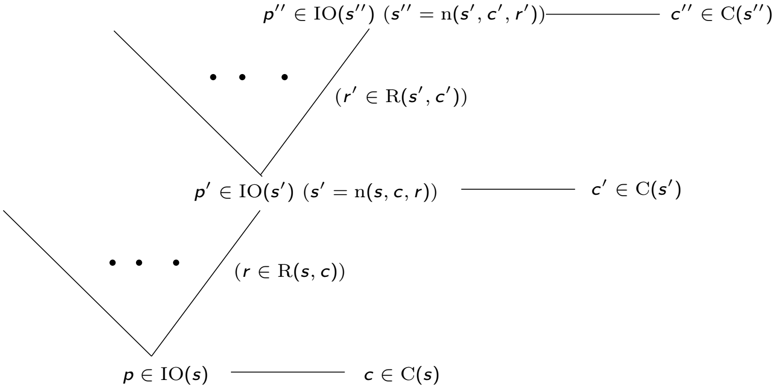
Need for Coinductive Proofs

- In the beginning of computing, computer programs were batch programs.
  - One input one output
  - Correct programs correspond to **well-founded** structures (termination).
- Nowadays most programs are interactive;
  - A possibly infinite sequence of interactions, often concurrently.
  - Correspond to **non-well-founded** structures.
  - For instance non-concurrent computations can be represented as **IO-trees**.
  - A simple form of objects in object-oriented programs can be represented as non-well-founded trees.

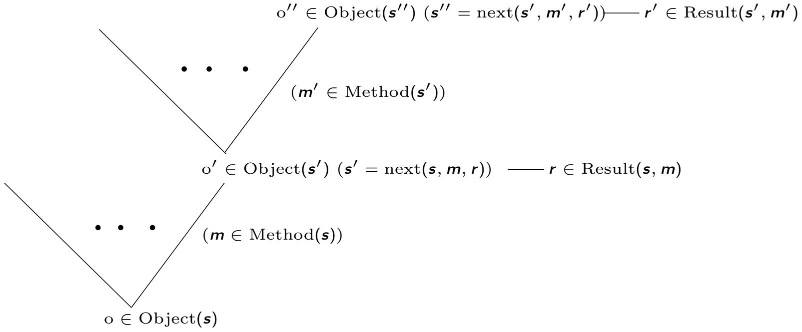
IO-Trees (Non-State Dependent)



IO-Trees State Dependent



Objects (State Dependent)

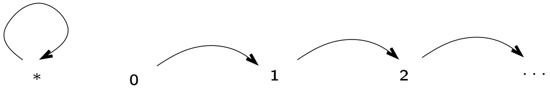


Need for Good Framework for Coinductive Structures

- Non-well-founded trees are defined coinductively.
- Relations between coinductive structures are coinductively defined
- Need suitable notion of reasoning coinductively.

Coinductive Proofs

- Reasoning about bisimulation is often very formalist. Consider an unlabelled Transition system:

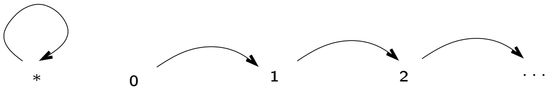


- For showing $* \sim n$ one defines
  - $R := \{(*, n) \mid n \in \mathbb{N}\}$
  - Shows that R is a bisimulation relation:
    - Let $(a, b) \in R$. Then $a = *$, $b = n \in \mathbb{N}$ for some n.
    - Assume $a = * \longrightarrow a'$.
      Then $a' = *$. We have $b = n \longrightarrow n + 1$ and $(*, n + 1) \in R$.
    - Assume $b = n \longrightarrow b'$.
      Then $b' = n + 1$. We have $a = * \longrightarrow *$ and $(*, n + 1) \in R$.
  - Therefore $x \sim y$ for $(x, y) \in R$.

Comparison

- Above is similar when carrying an inductive proof, e.g. of $\varphi := \forall n, m, k.(n+m)+k = n+(m+k)$ to defining
$$A := \{k \mid (n+m)+k = n+(m+k)\}$$
and showing that A is closed under 0 and successor.
- Instead we prove φ by induction on k using in the successor case the IH.
- Both proofs amount the same, but the second one would be far more difficult to teach and cumbersome to use.

Desired Coinductive Proof



- We show $\forall n \in \mathbb{N}. * \sim n$ by coinduction on $\sim$.
  - Assume $* \longrightarrow x$. We need to find y s.t. $n \longrightarrow y$ and $x \sim y$. Choose $y = n + 1$. By **co-IH** $* \sim n + 1$.
  - Assume $n \longrightarrow y$. We need to find x s.t. $* \longrightarrow x$ and $x \sim y$. Choose $x = *$. By **co-IH** $* \sim n + 1$.
- In essence same proof, but hopefully easier to teach and use.

Desired Coinductive Proof for Streams

- Consider $\mathrm{Stream} : \mathrm{Set}$ given coinductively by

$$\begin{array}{lcll} \mathrm{head} & : & \mathrm{Stream} \to \mathbb{N} & , \\ \mathrm{tail} & : & \mathrm{Stream} \to \mathrm{Stream} & . \end{array}$$

- Consider 3 versions of the stream $n, n+1, n+2, \ldots$

$$\begin{array}{lclclcl} \multicolumn{7}{l}{\mathrm{inc}, \mathrm{inc}', \mathrm{inc}'' : \mathbb{N} \to \mathrm{Stream}} \\ \mathrm{head}(\mathrm{inc}(n)) & = & \mathrm{head}(\mathrm{inc}'(n)) & = & \mathrm{head}(\mathrm{inc}''(n)) & = & n \\ \mathrm{tail}(\mathrm{inc}(n)) & = & \mathrm{inc}(n+1) \\ \mathrm{tail}(\mathrm{inc}'(n)) & = & \mathrm{inc}''(n+1) \\ \mathrm{tail}(\mathrm{inc}''(n)) & = & \mathrm{inc}'(n+1) \end{array}$$

Desired Coinductive Proof for Streams

- We show

$$\forall n \in \mathbb{N}.\mathrm{inc}(n) = \mathrm{inc}'(n) \land \mathrm{inc}(n) = \mathrm{inc}''(n)$$

by coinduction on Stream.

  - $\mathrm{head}(\mathrm{inc}(n)) = n = \mathrm{head}(\mathrm{inc}'(n)) = \mathrm{head}(\mathrm{inc}''(n))$
  - $\mathrm{tail}(\mathrm{inc}(n)) = \mathrm{inc}(n+1) \stackrel{\mathrm{co-IH}}{=} \mathrm{inc}''(n+1) = \mathrm{tail}(\mathrm{inc}'(n))$
  - $\mathrm{tail}(\mathrm{inc}(n)) = \mathrm{inc}(n+1) \stackrel{\mathrm{co-IH}}{=} \mathrm{inc}'(n+1) = \mathrm{tail}(\mathrm{inc}''(n))$

Goal

- Identify the precised dual of iteration, primitive recursion, induction.
- Identify the correct use of co-IH.
- Use of coalgebras as defined by their elimination rules.
- Generalise to indexed coinductively defined sets.

Introduction/Elimination of Inductive/Coinductive Sets

- Introduction rules for Natural numbers means that we have

$$\begin{array}{l} 0 \in \mathbb{N} \\ \mathrm{S} : \mathbb{N} \to \mathbb{N} \end{array}$$

  so we have an $\mathbb{N}$-algebra

$$(\mathbb{N}, 0, \mathrm{S}) \in (X \in \mathrm{Set}) \times X \times (X \to X)$$

- Dually, coinductive sets are given by their elimination rules i.e. by **observations** or **eliminators**.
  As an example we consider Stream:

$$\begin{array}{lcl} \mathrm{head} & : & \mathrm{Stream} \to \mathbb{N} \\ \mathrm{tail} & : & \mathrm{Stream} \to \mathrm{Stream} \end{array}$$

  We obtain a Stream-coalgebra

$$(\mathrm{Stream}, \mathrm{head}, \mathrm{tail}) \in (X \in \mathrm{Set}) \times (X \to \mathbb{N}) \times (X \to X)$$

Problem of Defining Coalgebras by their Introduction Rules

- Commonly one defines coalgebras by their introduction rules:
  Stream is the largest set closed under

$$\mathrm{cons} : \mathrm{Stream} \times \mathbb{N} \to \mathrm{Stream}$$

- Problem:
  - In **set theory** cons cannot be defined as a constructor such as

$$\mathrm{cons}(n, s) := \langle \lceil \mathrm{cons} \rceil, n, s \rangle$$

    as for inductively defined sets, since we would need
    non-well-founded sets.
    We can define a set Stream closed under a function cons, but that's no longer the same operation one would use for defining a corresponding inductively defined set.
  - In a **term model** we obtain **non-normalisation**:
    We get elements such as

$$\mathrm{zerostream} := \mathrm{cons}(0, \mathrm{cons}(0, \mathrm{cons}(0, \cdots))) \in \mathrm{Stream}$$

Problem of Defining Coalgebras by their Introduction Rules

- If we define Stream by its elimination rules, problems vanish:
  - In set theory Stream is a set which allows operations $\mathrm{head} : \mathrm{Stream} \to \mathbb{N}$, $\mathrm{tail} : \mathrm{Stream} \to \mathrm{Set}$.
    For instance we can take
    $$\begin{array}{lcl} \mathrm{Stream} & := & \mathbb{N} \to \mathbb{N} \\ \mathrm{head}(f) & := & f(0) \\ \mathrm{tail}(f) & := & f \circ \mathrm{S} \end{array}$$
    and obtain a largest set in the sense given below.
  - In a term model we can define the streams as the largest set which allows to define head and tail.
    $\mathrm{zerostream}$ can be a term such that $\mathrm{head}(\mathrm{zerostream}) \longrightarrow 0$, $\mathrm{tail}(\mathrm{zerostream}) \longrightarrow \mathrm{zerostream}$.
    $\mathrm{zerostream}$ itself is in normal form.
- In both cases cons can now be **defined** by the principle of coiteration.

Unique Iteration

- That $(\mathbb{N}, 0, \mathrm{S})$ are minimal can be given by:
  - Assume another $\mathbb{N}$-algebra (X, z, s), i.e.

$$\begin{array}{l} z \in X \\ s : X \to X \end{array}$$

  - Then there exist a **unique homomorphism** $g : (\mathbb{N}, 0, \mathrm{S}) \to (X, z, s)$, i.e.

$$\begin{array}{lcl} g : \mathbb{N} \to X & & \\ g(0) & = & z \\ g(\mathrm{S}(n)) & = & s(g(n)) \end{array}$$

  - This is the same as saying $\mathbb{N}$ is an initial $\mathrm{F}_{\mathbb{N}}$-algebra.
  - This means we can define uniquely

$$\begin{array}{lcll} g : \mathbb{N} \to X & & & \\ g(0) & = & x & \text{for some } x \in X \\ g(\mathrm{S}(n)) & = & x' & \text{for some } x' \in X \text{ depending on } g(n) \end{array}$$

  - This is the principle of **unique iteration**.
  - Definition by **pattern matching**.

Unique Coiteration

- Dually, that $(\mathrm{Stream}, \mathrm{head}, \mathrm{tail})$ is maximal can be given by:
  - Assume another Stream-coalgebra (X, h, t):

$$\begin{array}{rcl} h & : & X \to \mathbb{N} \\ t & : & X \to X \end{array}$$

  - Then there exist a **unique homomorphism** $g : (X, h, t) \to (\mathrm{Stream}, \mathrm{head}, \mathrm{tail})$, i.e.:

$$\begin{array}{lcl} g : X \to \mathrm{Stream} & & \\ \mathrm{head}(g(x)) & = & h(x) \\ \mathrm{tail}(g(x)) & = & g(t(x)) \end{array}$$

- Means we can define uniquely

$$\begin{array}{lcll} g : X \to \mathrm{Stream} & & & \\ \mathrm{head}(g(x)) & = & n & \text{for some } n \in \mathbb{N} \text{ depending on } x \\ \mathrm{tail}(g(x)) & = & g(x') & \text{for some } x' \in X \text{ depending on } x \end{array}$$

  This is the principle of **unique coiteration**.
- Definition by **copattern matching**.

Comparison

- When using iteration the instance of g we can use is restricted, but we can apply an arbitrary function to it.
- When using coiteration we can choose any instance a of g, but cannot apply any function to $g(a)$.

Duality

[1]

Inductive Definition	Coinductive Definition
Determined by Introduction	Determined by Observation/Elimination
Iteration	Coiteration
Pattern matching	Copattern matching
Primitive Recursion	?
Induction	?
Induction-Hypothesis	?

[1] Part of this table is due to Peter Hancock, see acknowledgements at the end.

Unique Primitive Recursion

- From unique iteration for $\mathbb{N}$ we can derive principle of **unique primitive recursion**
  - We can define uniquely

$$
\begin{array}{lcll}
g : \mathbb{N} \to X & & & \\
g(0) & = & x & \text{for some } x \in X \\
g(\mathrm{S}(n)) & = & x' & \text{for some } x' \in X \text{ depending on } n,\ g(n)
\end{array}
$$

Unique Primitive Corecursion

- From unique coiteration we can derive principle of **unique primitive corecursion**
  - We can define uniquely

$$
\begin{array}{lcl}
g : X \to \mathrm{Stream} & & \\
\mathrm{head}(g(x)) & = & n \text{ for some } n \in \mathbb{N} \text{ depending on } x \\
\mathrm{tail}(g(x))) & = & g(x') \text{ for some } x' \in X \text{ depending on } x \\
 & & \text{or} \\
 & = & s \text{ for some } s \in \mathrm{Stream} \text{ depending on } x
\end{array}
$$

Duality

- For primitive recursion we could make use of the pair $(n, g(n))$ consisting of n and the IH, i.e. an element of

$$\mathbb{N} \times X$$

- For primitive corecursion we can make use of either $s \in \mathrm{Stream}$ or $g(x')$, i.e. of an element of

$$\mathrm{Stream} + X$$

- $+$ is the dual of $\times$.

Duality

Inductive Definition	Coinductive Definition
Determined by Introduction	Determined by Observation/Elimination
Iteration	Coiteration
Pattern matching	Copattern matching
Primitive Recursion	Primitive Corecursion
Induction	?
Induction-Hypothesis	?

Example

$$s \in \mathrm{Stream}$$
$$\begin{array}{lcl} \mathrm{head}(s) & = & 0 \\ \mathrm{tail}(s) & = & s \end{array}$$

$$s' : \mathbb{N} \to \mathrm{Stream}$$
$$\begin{array}{lcl} \mathrm{head}(s'(n)) & = & 0 \\ \mathrm{tail}(s'(n)) & = & s'(n+1) \end{array}$$

$$\mathrm{cons} : (\mathbb{N} \times \mathrm{Stream}) \to \mathrm{Stream}$$
$$\begin{array}{lcl} \mathrm{head}(\mathrm{cons}(n, s)) & = & n \\ \mathrm{tail}(\mathrm{cons}(n, s)) & = & s \end{array}$$

Induction

- From unique iteration one can derive principle of **induction**:

  We can prove $\forall n \in \mathbb{N}.\varphi(n)$ by proving
  $\varphi(0)$
  $\forall n \in \mathbb{N}.\varphi(n) \rightarrow \varphi(\mathrm{S}(n))$

- Using induction we can prove (assuming extensionality of functions) uniqueness of iteration and primitive recursion.

Equivalence

Theorem

Let $(\mathbb{N}, 0, \mathrm{S})$ *be an* $\mathbb{N}$*-algebra. The following is equivalent*

1. *The principle of unique iteration.*
2. *The principle of unique primitive recursion.*
3. *The principle of iteration + induction.*
4. *The principle of primitive recursion + induction.*

Coinduction

- Uniqueness in coiteration is equivalent to the principle:
  Bisimulation implies equality
- Bisimulation on Stream is the largest relation $\sim$ on Stream s.t.

$$s \sim s' \to \mathrm{head}(s) = \mathrm{head}(s') \land \mathrm{tail}(s) \sim \mathrm{tail}(s')$$

- Largest can be expressed as $\sim$ being an indexed coinductively defined set.
- Primitive corecursion over $\sim$ means:
  We can prove

$$\forall s, s'.X(s, s') \to s \sim s'$$

  by showing

$$\begin{array}{rcl} X(s, s') & \to & \mathrm{head}(s) = \mathrm{head}(s') \\ X(s, s') & \to & X(\mathrm{tail}(s), \mathrm{tail}(s')) \lor \mathrm{tail}(s) \sim \mathrm{tail}(s') \end{array}$$

Coinduction

- Combining
  - bisimulation implies equality
  - bisimulation can be shown corecursively

  we obtain the following principle of **coinduction**

Schema of Coinduction

- We can prove
$$\forall s, s'.X(s,s') \to s = s'$$
by showing
$$\begin{array}{lcl} \forall s, s'.X(s,s') & \to & \mathrm{head}(s) = \mathrm{head}(s') \\ \forall s, s'.X(s,s') & \to & \mathrm{tail}(s) = \mathrm{tail}(s') \end{array}$$
where $\mathrm{tail}(s) = \mathrm{tail}(s')$ can be derived
  - directly or
  - from a proof of
  $$X(\mathrm{tail}(s), \mathrm{tail}(s'))$$
  invoking the **co-induction-hypothesis**
  $$X(\mathrm{tail}(s), \mathrm{tail}(s')) \to \mathrm{tail}(s) = \mathrm{tail}(s')$$
- **Note:** Only direct use of co-IH allowed.

Equivalence

Theorem

Let $(\mathrm{Stream}, \mathrm{head}, \mathrm{tail})$ *be a* Stream*-coalgebra. The following is equivalent*

1. *The principle of unique coiteration.*
2. *The principle of unique primitive corecursion.*
3. *The principle of coiteration + coinduction*
4. *The principle of primitive corecursion + coinduction*

Duality

Inductive Definition	Coinductive Definition
Determined by Introduction	Determined by Observation/Elimination
Iteration	Coiteration
Pattern matching	Copattern matching
Primitive Recursion	Primitive Corecursion
Induction	Coinduction
Induction-Hypothesis	Coinduction-Hypothesis

General Strictly Positive Indexed Inductive Definitions

- Strictly positive indexed inductively defined sets over index set I are collection of sets $\mathrm{D} : \mathrm{I} \to \mathrm{Set}$ closed under constructors

$$\begin{aligned}\mathrm{C}_j : & (x_1 \in A_1) \times (x_2 \in A_2(x_1)) \times \cdots \times (x_n \in A_n(x_1, \ldots, x_{n-1})) \\ & \to \mathrm{D}(\mathrm{i}(x_1, \ldots, x_n))\end{aligned}$$

- Here $A_k(\vec{x})$ is
  - either a non-inductive argument, i.e. a set independent of A,
  - or it is an inductive argument, i.e.

$$A_k(\vec{x}) = (b \in B(\vec{x})) \to \mathrm{D}(\mathrm{i}'_k(\vec{x}, b))$$

- Later arguments **cannot depend** on inductive arguments, only on non-inductive arguments.

Simplification

- Therefore we can move the inductive arguments to the end $(\vec{x} := x_1, \ldots, x_k)$

$$
\begin{array}{l}
\mathrm{C}_j : \underbrace{(x_1 \in A_1) \times (x_2 \in A_2(x_1)) \times \cdots \times x_k \in A_k(x_1, \ldots, x_{k-1})}_{\text{non-inductive arguments}} \times \\
\quad\underbrace{(b \in B_1(\vec{x})) \to \mathrm{D}(\mathrm{i}'_1(\vec{x}, b)) \times \cdots \times (b \in B_l(\vec{x})) \to \mathrm{D}(\mathrm{i}'_l(\vec{x}, b)))}_{\text{inductive arguments}} \\
\quad\to \mathrm{D}(\mathrm{i}_j(\vec{x}))
\end{array}
$$

Simplification

$$\mathrm{C}_j : \underbrace{(x_1 \in A_1) \times (x_2 \in A_2(x_1)) \times \cdots \times x_k \in A_k(x_1, \ldots, x_{k-1})}_{\text{non-inductive arguments}} \times$$
$$\underbrace{(b \in B_1(\vec{x})) \to \mathrm{D}(\mathrm{i}'_1(\vec{x}, b)) \times \cdots \times (b \in B_l(\vec{x})) \to \mathrm{D}(\mathrm{i}'_l(\vec{x}, b))}_{\text{inductive arguments}})$$
$$\to A(\mathrm{i}_j(\vec{x}))$$

- We can form now the product of the non-inductive arguments and obtain a single non-inductive argument.
- We can replace the inductive arguments by one non-inductive argument

$$(b \in (B_1(\vec{x}) + \cdots + B_l(\vec{x}))) \to \mathrm{D}(\mathrm{i}''(\vec{x}, b))$$

for some i''.

Simplification

- We obtain for some new sets $A_j, B_j(x)$ and function j, i

$$\mathrm{C}_j : ((a \in A_j) \times ((b \in B_j(a)) \to \mathrm{D}(\mathrm{j}(a,b)))) \to \mathrm{D}(\mathrm{i}(a))$$

- We can replace all constructors $\mathrm{C}_1, \ldots, C_n$ by one constructor C by adding an additional argument $j \in \{1, \ldots, n\}$ selecting the constructor, and then combine it with the non-inductive argument.
- So we have one constructor

$$\mathrm{C} : ((a \in A) \times ((b \in B(a)) \to \mathrm{D}(\mathrm{j}(a,b)))) \to \mathrm{D}(\mathrm{i}(a))$$

Restricted Indexed (Co)Inductively Defined Sets

$$\mathrm{C} : ((a \in A) \times ((b \in B(a)) \to \mathrm{D}(\mathrm{j}(a, b)))) \to \mathrm{D}(\mathrm{i}(a))$$

- In order to obtain the corresponding observations/eliminators for the corresponding co-inductive definitions, we need to invert the arrows.
- The more natural dual is obtained if we use restricted indexed inductive definitions:

$$\mathrm{C} : (i \in \mathrm{I}) \to ((a \in \mathrm{A}(i)) \times ((b \in \mathrm{B}(i, a)) \to \mathrm{D}(\mathrm{j}(i, a, b)))) \to \mathrm{D}(i)$$

- The corresponding observations/eliminators are

$$\mathrm{E} : (i \in \mathrm{I}) \to \mathrm{D}(i) \to ((a \in \mathrm{A}(i)) \times ((b \in \mathrm{B}(i, a)) \to \mathrm{D}(\mathrm{j}(i, a, b))))$$

or

$$\mathrm{E} : ((i \in \mathrm{I}) \times \mathrm{D}(i)) \to ((a \in \mathrm{A}(i)) \times ((b \in \mathrm{B}(i, a)) \to \mathrm{D}(\mathrm{j}(i, a, b))))$$

Petersson-Synek Trees

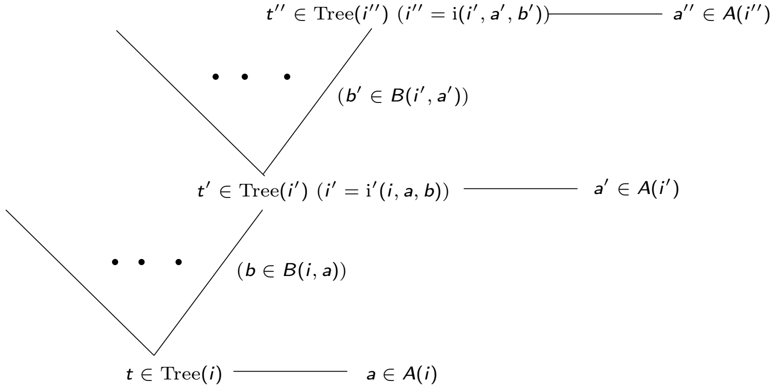
- $\mathrm{D}(i)$ form the Petersson-Synek trees (observation by Hancock), which correspond as well to the containers by Abbott, Altenkirch and Ghani.
- Replacing D by the more meaningful name Tree we obtain

$$
\begin{array}{l}
\text{data } \mathrm{Tree} : \mathrm{I} \to \mathrm{Set} \text{ where} \\
\quad \mathrm{C} : ((i \in \mathrm{I}) \times \\
\qquad (a \in \mathrm{A}(i)) \times ((b \in \mathrm{B}(i,a)) \to \mathrm{Tree}(\mathrm{j}(i,a,b)))) \\
\qquad \to \mathrm{Tree}(i)
\end{array}
$$

- For the corresponding coinductive defined set Tree^∞ we divide E into its two components and obtain

$$
\begin{array}{lcl}
\multicolumn{3}{l}{\text{coalg } \mathrm{Tree}^\infty : \mathrm{I} \to \mathrm{Set} \text{ where}} \\
\mathrm{E}_1 & : & ((i \in \mathrm{I}) \times \mathrm{Tree}^\infty(i)) \to \mathrm{A}(i) \\
\mathrm{E}_2 & : & ((i \in \mathrm{I}) \times (t \in \mathrm{Tree}^\infty(i)) \times (b \in \mathrm{B}(i, \mathrm{E}_1(i,t)))) \\
 & & \quad \to \mathrm{Tree}^\infty(\mathrm{j}(i, \mathrm{E}_1(i,t), b))
\end{array}
$$

Petersson-Synek Trees



Equivalence of unique (Co)induction, (Co)recursion, (Co)induction

- The notions of (co)iteration, primitive (co)recursion, (co)induction can be generalised in a straightforward way to Petersson-Synek Trees and Co-Trees.
- One can show the equivalence of
  - unique iteration, unique primitive recursion, iteration + induction, primitive recursion + induction
  - unique coiteration, unique primitive corecursion, coiteration + coinduction, primitive corecursion + coinduction
- We call Petersson-Synek algebras fulfilling unique iteration initial Petersson-Synek algebras.
- We call Petersson-Synek coalgebras fulfilling unique coiteration final Petersson-Synek coalgebras.

Concrete Model of Tree^∞

- Tree can be modelled in a straightforward way set theoretically.
- A very concrete model of Tree^∞ can be defined by following the principle that a coalgebra is given by its observations.
  - The result of E_1 can be observed directly.
  - The result of E_2 is an element of $\mathrm{Tree}^\infty(i')$ for some i' which can be observed by carrying out more observations.

Concrete Model of Tree^∞

- Let for $i \in \mathrm{I}$

$$
\begin{aligned}
\mathrm{Path}_{[\![\,\mathrm{Tree}^\infty\,]\!]}(i) := \{\langle i_0, a_0, b_0, i_1, a_1, b_1, \ldots, i_n, a_n\rangle \mid\ & \\
& n \geq 0, i_0 = i, \\
& (\forall k \in \{0, \ldots, n-1\}.b_k \in \mathrm{B}(i_k, a_k) \wedge \\
& \qquad\qquad i_{k+1} = \mathrm{j}(i_k, a_k, b_k)), \\
& \forall k \in \{0, \ldots, n\}.a_k \in \mathrm{A}(i_k)\}
\end{aligned}
$$

- Let $[\![\,\mathrm{Tree}^\infty\,]\!](i)$ be the set of $t \subseteq \mathrm{Path}_{[\![\,\mathrm{Tree}^\infty\,]\!]}(i)$ which form the set of paths of a tree:
  - $\langle i_0, a_0, b_0, \ldots, i_{n+1}, a_{n+1}\rangle \in t \to \langle i_0, a_0, b_0, \ldots, i_n, a_n\rangle \in t$
  - $\exists! a.\langle i, a\rangle \in t$,
  - $\langle i_0, a_0, b_0, \ldots, i_n, a_n\rangle \in t \wedge b_n \in \mathrm{B}(i_n, a_n) \wedge i_{n+1} = \mathrm{j}(i_n, a_n, b_n)$
    $\to \exists! a_{n+1}.\langle i_0, a_0, b_0, \ldots, i_n, a_n, b_n, i_{n+1}, a_{n+1}\rangle \in t$

Concrete Model of Tree^∞

- Define

$$\begin{array}{l} \mathrm{E}_1 : (i \in \mathrm{I}) \to [\![\, \mathrm{Tree}^\infty \,]\!](i) \to \mathrm{A}(i) \\ \mathrm{E}_1(i, t) := a \qquad \text{if } \langle i, a \rangle \in t \end{array}$$

- Define

$$\begin{array}{l} \mathrm{E}_2 : ((i \in \mathrm{I}) \to (t \in [\![\, \mathrm{Tree}^\infty \,]\!](i)) \to (b \in \mathrm{B}(i, \mathrm{E}_1(i, t))) \\ \qquad \to [\![\, \mathrm{Tree}^\infty \,]\!](\mathrm{j}(i, \mathrm{E}_1(i, t), b)) \\ \mathrm{E}_2(i, t, b) := \{\langle i_1, a_1, b_1, \ldots, i_{n+1}, a_{n+1} \rangle \\ \qquad\qquad\qquad \mid \langle i, \mathrm{E}_1(i, t), b, i_1, a_1, b_1, \ldots, i_{n+1}, a_{n+1} \rangle \in t\} \end{array}$$

Concrete Model of Tree^∞

Theorem

$(\llbracket \mathrm{Tree}^\infty \rrbracket, \mathrm{E}_1, \mathrm{E}_2)$ *is a final* Tree^∞*-coalgebra.*

Schema for Primitive Corecursion

- Assume $A : \mathrm{I} \to \mathrm{Set}$, $[\![\, \mathrm{Tree}^\infty \,]\!], \mathrm{E}_1, \mathrm{E}_2$ as before. We can define a function
$$f : (i \in \mathrm{I}) \to X(i) \to [\![\, \mathrm{Tree}^\infty \,]\!](i)$$
corecursively by defining for $i \in \mathrm{I}$, $x \in X(i)$
  - a value $a' := \mathrm{E}_1(i, f(i, x)) \in \mathrm{A}(i)$
  - and for $b \in \mathrm{B}(i, a)$ a value $\mathrm{E}_2(i, f(i, x), b) \in [\![\, \mathrm{Tree}^\infty \,]\!](i', b)$
    where $i' := \mathrm{j}(i, a', b)$
    and we can define $\mathrm{E}_2(i, f(i, x), b)$
    - as an element of $[\![\, \mathrm{Tree}^\infty \,]\!](i')$ defined before
    - or corecursively define $\mathrm{E}_2(i, f(i, x), b) = f(i', x')$
      for some $x' \in X(i')$.
      Here $f(i', x')$ will be called the corecursion hypothesis.

Example

- Define the set of increasing streams $\mathrm{IncStream} : \mathbb{N} \to \mathrm{Set}$ starting with at least n coinductively by

$$\begin{array}{lcl} \mathrm{head} & : & (n : \mathbb{N}) \to \mathrm{IncStream}(n) \to \mathbb{N}_{\geq n} \\ \mathrm{tail} & : & (n : \mathbb{N}) \to (s : \mathrm{IncStream}(n)) \to \mathrm{IncStream}(\mathrm{head}(n, s) + 1) \end{array}$$

where $\mathbb{N}_{\geq n} := \{m : \mathbb{N} \mid m \geq n\}$.
Define

$$\begin{array}{lcl} \mathrm{inc}, \mathrm{inc}', \mathrm{inc}'' & : & (n : \mathbb{N}) \to \mathrm{IncStream}(n) \\ \mathrm{head}(n, \mathrm{inc}(n)) & = & \mathrm{head}(n, \mathrm{inc}'(n)) = \mathrm{head}(n, \mathrm{inc}''(n)) = n \\ \mathrm{tail}(n, \mathrm{inc}(n)) & = & \mathrm{inc}(n + 1) \\ \mathrm{tail}(n, \mathrm{inc}'(n)) & = & \mathrm{inc}''(n + 1) \\ \mathrm{tail}(n, \mathrm{inc}''(n)) & = & \mathrm{inc}'(n + 1) \end{array}$$

Schema for Indexed Corecursively Defined Functions

- Assume $X \in \mathrm{Set}$, $\widehat{j} : X \to \mathrm{I}$.
  We can define
  $$f : (x \in X) \to [\![\, \mathrm{Tree}^{\infty} \,]\!](\widehat{j}(x))$$
  corecursively by determining for $x \in X$ with $i := \widehat{j}(x)$,
  - $a := \mathrm{E}_1(i, f(x)) \in \mathrm{A}(i)$
  - and for $b \in \mathrm{B}(i, a)$ with $i' := \mathrm{j}(i, a, b)$ the value $\mathrm{E}_2(i, f(x), b) \in [\![\, \mathrm{Tree}^{\infty} \,]\!](i')$
    where we can define $\mathrm{E}_2(i, f(x), b)$ as
    - a previously defined value of $[\![\, \mathrm{Tree}^{\infty} \,]\!](i')$
    - or corecursively define $\mathrm{E}_2(i, f(x), b) = f(x')$ for some x' such that $\widehat{j}(x') = i'$.
      $f(x')$ will be called the corecursion hypothesis.

Example

- Define $\mathrm{Stack} : \mathbb{N} \to \mathrm{Set}$ coinductively with destructors

$$\begin{array}{lcl} \mathrm{top} & : & ((n \in \mathbb{N}) \times (n > 0) \times \mathrm{Stack}(n)) \to \mathbb{N} \\ \mathrm{pop} & : & ((n \in \mathbb{N}) \times (n > 0) \times \mathrm{Stack}(n)) \to \mathrm{Stack}(n-1) \end{array}$$

- We can define $\mathrm{empty} : \mathrm{Stack}(0)$, where we do not need to define anything since $0 > 0 = \emptyset$.
- We can define

$$\begin{array}{lcl} \multicolumn{3}{l}{\mathrm{push} : (n, m \in \mathbb{N}) \to \mathrm{Stack}(n) \to \mathrm{Stack}(n+1) \qquad \text{s.t.}} \\ \mathrm{top}(n+1, *, \mathrm{push}(n, m, s)) & = & m \\ \mathrm{pop}(n+1, *, \mathrm{push}(n, m, s)) & = & s \end{array}$$

Schema for Coinduction

- Assume

$$\begin{array}{lcl} J & : & \mathrm{Set} \\ \widehat{i} & : & J \to \mathrm{I} \\ x_0, x_1 & : & (j \in J) \to [\![\, \mathrm{Tree}^{\infty} \,]\!](\widehat{i}(j)) \end{array}$$

  We can show $\forall j \in J.x_0(j) = x_0(j')$ coinductively by showing
  - $\mathrm{E}_0(\widehat{i}(j), x_0(j))$ and $\mathrm{E}_0(\widehat{i}(j), x_1(j))$ are equal
  - and for all b that
    $\mathrm{E}_1(\widehat{i}(j), x_0(j), b)$ and $\mathrm{E}_1(\widehat{i}(j), x_0(j), b)$ are equal,
    where we can use either the fact that
    - this was shown before,
    - or we can use the coinduction-hypothesis, which means using the fact $\mathrm{E}_1(\widehat{i}(j), x_0(j), b) = x_0(j')$ and $\mathrm{E}_1(\widehat{i}(j), x_1(j), b) = x_1(j')$ for some $j' \in J$.

Example

- Let

$$\begin{array}{rcl} s \in \mathrm{Stream} & & \\ \mathrm{head}(s) & = & 0 \\ \mathrm{tail}(s) & = & s \end{array} \qquad \begin{array}{rcl} s' : \mathbb{N} \to \mathrm{Stream} & & \\ \mathrm{head}(s'(n)) & = & 0 \\ \mathrm{tail}(s'(n)) & = & s'(n+1) \end{array}$$

$$\begin{array}{rcl} \mathrm{cons} : \mathbb{N} \to \mathrm{Stream} \to \mathrm{Stream} & & \\ \mathrm{head}(\mathrm{cons}(n,s)) & = & n \\ \mathrm{tail}(\mathrm{cons}(n,s)) & = & s \end{array}$$

- We show $\forall n \in \mathbb{N}.s = s'(n)$ by coinduction:
  Assume $n \in \mathbb{N}$. $\mathrm{head}(s) = \mathrm{head}(s'(n))$ and $\mathrm{tail}(s) = s = s'(n+1) = \mathrm{tail}(s'(n))$, where $s = s'(n+1)$ follows by the coinduction hypothesis.
- We show $\mathrm{cons}(0,s) = s$ by coinduction:
  $\mathrm{head}(\mathrm{cons}(0,s)) = 0 = \mathrm{head}(s)$ and $\mathrm{tail}(\mathrm{cons}(0,s)) = s = \mathrm{tail}(s)$, where we did not use the coinduction hypothesis.

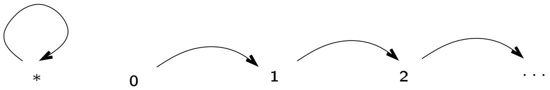
Schema for Bisimulation on Labelled Transition Systems

- Bisimulation is an indexed coinductively defined relation and therefore proofs of bisimulation can be shown by corecursion.
- Assume a labelled transition system with states S, labels L and a relation $\longrightarrow \subseteq \mathrm{S} \times \mathrm{L} \times \mathrm{S}$

Schema for Bisimulation on Labelled Transition Systems

- Let $I \in \mathrm{Set}$, $s, s' : I \to \mathrm{S}$.
- We can prove $\forall i \in \mathrm{I}.\mathrm{Bisim}(s(i), s'(i))$ coinductively by defining for any $i \in \mathrm{I}$
  - for any $l \in \mathrm{L}$, $s_0 \in \mathrm{S}$ s.t. $s(i) \xrightarrow{l} s_0$ an $s_0' \in \mathrm{S}$ s.t.
    - $s'(i) \xrightarrow{l} s_0'$
    - and s.t. $\mathrm{Bisim}(s_0, s_0')$
      where one can for prove the latter by invoking the Coinduction Hypothesis
      $\mathrm{Bisim}(s(i'), s'(i'))$ for some i' such that $s(i') = s_0$, $s'(i') = s_0'$.
  - for any $l \in \mathrm{L}$, $s_0' \in \mathrm{S}$ s.t. $s'(i) \xrightarrow{l} s_0'$ an $s_0 \in \mathrm{S}$ s.t.
    - $s(i) \xrightarrow{l} s_0$
    - and s.t. $\mathrm{Bisim}(s_0, s_0')$
      where one can prove the latter by invoking the Coinduction Hypothesis
      $\mathrm{Bisim}(s(i'), s'(i'))$ for some i' such that $s(i') = s_0$, $s'(i') = s_0'$.

Example from Introduction



- We show $\forall n \in \mathbb{N}. * \sim n$ by coinduction on $\sim$.
  - Assume $* \longrightarrow x$. We need to find y s.t. $n \longrightarrow y$ and $x \sim y$. Choose $y = n + 1$. By **co-IH** $* \sim n + 1$.
  - Assume $n \longrightarrow y$. We need to find x s.t. $* \longrightarrow x$ and $x \sim y$. Choose $x = *$. By **co-IH** $* \sim n + 1$.
- In essence same proof, but hopefully easier to teach and use.

Generalisation

- The previous example can be generalised to arbitrary coinductively defined relations.

Object-Oriented/Based Programming

- Object-oriented (OO) programming is currently main programming paradigm.
- OO programming has lots of components, we will here only look at the notion of an object.
- The component of OO programming dealing with objects only is called **object-based programming**.

Example: cell in Java

```
class cell <A> {

    /* Instance Variable */
    A content;

    /* Constructor */
    cell (A s) { content = s; }

    /* Method put */
    public void put (A s) { content = s; }

    /* Method get */
    public A get () { return content; }
}
```

Modelling Methods as Objects

- The type of objects cell is modelled as
  - a coalgebra Cell
  - with observations being the methods.
- A method m with argument A and return type B is modelled as observation
$$\mathsf{m} : \mathsf{Cell} \to \mathsf{A} \to \mathsf{B} \times \mathsf{Cell}$$
which for a cell and an argument A returns the return type and the updated cell.
- Return type void is modelled as the one element type Unit.
- Access to instant variables is not needed, since we can use get and put for it.
- A constructor with argument A is modelled as a function defined by guarded recursion
$$\mathsf{cell} : \mathsf{A} \to \mathsf{Cell}$$

Object as a Coalgebra

We define the cell using the notation we would like to have:

$$
\begin{array}{l}
\text{coalg Cell } (\mathsf{A} : \mathrm{Set}) \text{ where} \\
\quad \begin{array}{lcllllllll}
\text{put} & : & \text{Cell A} & \rightarrow & \text{A} & \rightarrow & (\text{Unit} & \times & \text{Cell A}) \\
\text{get} & : & \text{Cell A} & \rightarrow & \text{Unit} & \rightarrow & (\text{A} & \times & \text{Cell A})
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{cell} : \{\mathsf{A} : \text{Set}\} \rightarrow \mathsf{A} \rightarrow \text{Cell A} \\
\begin{array}{llclcl}
\text{put} & (\text{cell a}) & \text{b} & = & (\text{unit} & , \ \text{cell b}) \\
\text{get} & (\text{cell a}) & \_ & = & (\text{a} & , \ \text{cell a})
\end{array}
\end{array}
$$

- {A : Set} denotes a hidden argument which can be omitted and inferred by the system.

Object as a Coalgebra

- Official Agda code uses record instead of coalg
  - in the fields one needs to add as argument the record being defined.
  - there is a bit more generality which results in more beaurocracy.

```
record Cell (X : Set) : Set where
  coinductive
  field
    put : X    → Unit × Cell X
    get : Unit → X    × Cell X


cell : {X : Set} → X → Cell X
put (cell x) y = (unit , cell y)
get (cell x) _ = (x    , cell x)
```

- More details see Kyoto Talk.

Conclusion

- **Coiteration, primitive corecursion, coinduction** are the **duals** of iteration, primitive recursion, induction.
- In iteration/recursion/induction,
  - the **instances** of the **IH** used are **restricted**,
  - the **result can be used** in **arbitrary** functions and formulas.
- In **coiteration/corecursion/coinduction**,
  - the **instances** of the co-IH are **unrestricted**,
  - but the **result** can be only used **only directly**.
- General case of **indexed coinductive definitions** can be **reduced** to **Petersson-Synek Cotrees**.

Conclusion

- **Schemata** for **primitive corecursion and coinduction**.
- Schemata can be applied to indexed coinductively defined sets and relations.
- **Relations** on coinductively defined sets seem to be often **coinductively defined indexed relations** and can be shown by **indexed corecursion** which can be regarded as **coinduction**.
- **Objects** as in OOprogramming are **naturally occurring coalgebras**.
- **Objects** are **determined** by their **observations** and can be defined in a **natural way** in **Agda**

References

- Most of this talk (on coalgebras) was based on [Set16].
- Article on coalgebras in Martin-Löf Type Theory [Set12].
- Copatterns: [APTS13, SAPT14].
- Objects in Martin-Löf Type Theory:
  - First article [Set07],
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Switch over to Talk by Bashar