Sec. 8: Semi-Computable Predicates

- We study $P \subseteq \mathbb{N}^n$, which are
  - **not decidable**,
  - but "**half decidable**".
- Official name is
  - **semi-decidable**,
  - or **semi-computable**.
  - or **recursively enumerable (r.e.)**.

Rec. Sets

Remember:

- A predicate A is recursive, iff χ_A is recursive.
- So we have a "full" decision procedure:

$$\begin{array}{rcl} P(\vec{x}) & \Leftrightarrow & \chi_P(\vec{x}) = 1, \text{ i.e. answer yes }, \\ \neg P(\vec{x}) & \Leftrightarrow & \chi_P(\vec{x}) = 0, \text{ i.e. answer no }. \end{array}$$

Rec.enum. vs. semi-decidable

- **Recursively enumerable** stands for the definition based on the notion of partial recursive functions.
- **Semi-decidable** or **semi-computable** stand for the definition based on an intuitive notion of "(partial) computable function"
- Assuming the **Church-Turing thesis**, the two notions coincide.

Semi-Decidable Sets

$P \subseteq \mathbb{N}^n$ will be semi-decidable,
if there exists a partial recursive recursive function f s.t.

$$P(\vec{x}) \Leftrightarrow f(\vec{x}) \downarrow \quad .$$

- If $P(\vec{x})$ holds,
  we will eventually know it:
  the algorithm for computing f will finally terminate,
  and then we know that $P(\vec{x})$ holds.
- If $P(\vec{x})$ doesn't hold,
  then the algorithm computing f will loop for ever,
  and we never get an answer.

Semi-Decidable Sets

So we have:

$$P(\vec{x}) \Leftrightarrow f(\vec{x})\downarrow \text{ i.e. answer yes },$$
$$\neg P(\vec{x}) \Leftrightarrow f(\vec{x})\uparrow \text{ i.e. no answer returned by } f .$$

Applications

- **Examples** (Cont.)
  - Type checking in Agda (used in the module interactive theorem proving) is semi-decidable.
    - Does in most applications not cause any problems

Jump over next example

Applications

- One might think that semi-computable sets don't occur in computing.
- But they occur in many applications.
- **Examples** are
  - Checking whether a program terminates is semi-decidable.
  - Checking whether a program in C++ is type correct is because of the template mechanism semi-decidable.
    - In C++ compilers this problem is usually prevented by having a flag which limits the number of times templates are unfolded.

Applications

- Whether a statement is provable in many logical systems is semi-decidable.
  - But even so this is semi-decidable, many search algorithm succeed in most practical cases.
  - Often one can predict a certain time, after which normally the search algorithm should have returned an answer.
    - If the search algorithm hasn't returned an answer after this time it is likely (but not guaranteed) that the statement is unprovable.

Def. 8.1 (Recursively Enumerable)

- A predicate $A \subseteq \mathbb{N}^n$ is **recursively enumerable**,
  in short **r.e.**,
  if there exists a partial recursive function $f : \mathbb{N}^n \xrightarrow{\sim} \mathbb{N}$ s.t.

$$A = \mathrm{dom}(f) \ \ .$$

- Sometimes recursive predicates are as well called
  - **semi-decidable** or
  - **semi-computable** or
  - **partially computable**.

Lemma 8.3

(a) Every recursive predicate is r.e.

(b) The **halting problem**, i.e.

$$\mathrm{Halt}^n(e, \vec{x}) :\Leftrightarrow \{e\}^n(\vec{x}) \downarrow \ ,$$

is r.e., but not recursive.

The proof of Lemma 8.3 and the statement and proof of Theorem 8.4 will be omitted in this lecture

Jump over proof of Lemma 8.3 and Theorem 8.4.

Proof of Lemma 8.3

(a)
- Assume $A \subseteq \mathbb{N}^k$ is decidable.
- Then

$$\mathbb{N}^k \setminus A$$

  is recursive, therefore its characteristic function

$$\chi_{\mathbb{N}^k \setminus A}$$

  is recursive as well.
- Define

$$f : \mathbb{N}^k \xrightarrow{\sim} \mathbb{N},\ f(\vec{x}) :\simeq (\mu y.\chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0) \ \ .$$

- Note that y doesn't occur in the body of the μ-expression.

Proof of Lemma 8.3

- Then we have
  - If $A(\vec{x})$, then

$$\chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0 \ \ ,$$

    so

$$f(\vec{x}) \simeq (\mu y.\chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0) \simeq 0 \ \ ,$$

    especially

$$f(\vec{x}) \downarrow \ \ .$$

Proof of Lemma 8.3

- If $(\mathbb{N}^k \setminus A)(\vec{x})$, then

$$\chi_{\mathbb{N}^k\setminus A}(\vec{x}) \simeq 1 \quad ,$$

so there exists no y s.t.

$$\chi_{\mathbb{N}^k\setminus A}(\vec{x}) \simeq 0 \quad .$$

therefore

$$f(\vec{x}) \simeq (\mu y.\chi_{\mathbb{N}^k\setminus A}(\vec{x}) \simeq 0) \simeq \bot \quad ,$$

especially

$$f(\vec{x})\uparrow \quad .$$

Proof of Lemma 8.3

(b)
- We have

$$\mathsf{Halt}^n(e,\vec{x}) :\Leftrightarrow f_n(e,\vec{x})\downarrow \quad ,$$

where f_n is partial recursive as in Sect. 5 s.t.

$$\{e\}^n(\vec{x}) \simeq f_n(e,\vec{x}) \quad .$$

- So

$$\mathsf{Halt}^n = \mathrm{dom}(f_n) \text{ is r.e.} \quad .$$

- We have seen above that Halt^n is non-computable, i.e. not recursive.

Jump over Theorem 8.4.

Proof of Lemma 8.3

- So we get

$$A(\vec{x}) \Leftrightarrow f(\vec{x})\downarrow \Leftrightarrow \vec{x} \in \mathrm{dom}(f) \quad ,$$

$$A = \mathrm{dom}(f) \text{ is r.e.} \quad .$$

Theorem 8.4

There exist r.e. predicates

$$\mathsf{W}^n \subseteq \mathbb{N}^{n+1}$$

s.t., with

$$\mathsf{W}^n_e := \{\vec{x} \in \mathbb{N}^n \mid \mathsf{W}^n(e,\vec{x})\} \quad ,$$

we have the following:

- Each of the predicates $\mathsf{W}^n_e \subseteq \mathbb{N}^n$ is r.e.
- For each r.e. predicate $P \subseteq \mathbb{N}^n$ there exists an $e \in \mathbb{N}$ s.t. $P = \mathsf{W}^n_e$, i.e.

$$\forall \vec{x} \in \mathbb{N}.P(\vec{x}) \Leftrightarrow \mathsf{W}^n_e(\vec{x}) \quad .$$

Theorem 8.4

Therefore, the r.e. sets $P \subseteq \mathbb{N}^n$ are exactly the sets W_e^n for $e \in \mathbb{N}$.

Proof Idea for Theorem 8.4

$$\mathrm{W}_e^n := \mathrm{dom}(\{e\}^n) \ .$$

If A is r.e., then $A = \mathrm{dom}(f)$ for some partial rec. f.
Let $f = \{e\}^n$.
Then $A = \mathrm{W}_e^n$.

The details given in the following will be omitted in the lecture. Jump over Details

Remark on Theorem 8.4

- W_e^n is therefore a **universal recursively enumerable sets**, which encodes all other recursively enumerable sets.
- The theorem means that that we can assign to every recursively enumerable predicate A a natural number, namely the e s.t. $A = \mathrm{W}_e^n$.
  - Each code denotes one predicate.
  - However, several numbers denote the same predicate:
    - there are e, e' s.t. $e \neq e'$, but $\mathrm{W}_e^n = \mathrm{W}_{e'}^n$. (Since there are $e \neq e'$ s.t. $\{e\}^n = \{e'\}^n$).

Proof of Theorem 8.4

- Let f_n s.t.
$$\forall e, \vec{n} \in \mathbb{N}. f_n(e, \vec{x}) \simeq \{e\}(\vec{x}) \ .$$
- Define
$$\mathrm{W}^n := \mathrm{dom}(f_n) \ .$$
- W^n is r.e.
- We have

$$\begin{aligned} \vec{x} \in \mathrm{W}_e^n &\Leftrightarrow (e, \vec{x}) \in \mathrm{W}^n \\ &\Leftrightarrow f_n(e, \vec{x}) \downarrow \\ &\Leftrightarrow \{e\}(\vec{x}) \downarrow \\ &\Leftrightarrow \vec{x} \in \mathrm{dom}(\{e\}^n) \ . \end{aligned}$$

Proof of Theorem 8.4

- Therefore
$$W^n_e = \mathrm{dom}(\{e\}^n) \ .$$
- W^n is r.e., since f_n is partial recursive.
- Furthermore, we have for any set $A \subseteq \mathbb{N}^n$

A is r.e. iff $A = \mathrm{dom}(f)$ for some partial recursive f
iff $A = \mathrm{dom}(\{e\}^n)$ for some $e \in \mathbb{N}$
iff $A = W^n_e$ for some $e \in \mathbb{N}$.

This shows the assertion.

Theorem 8.5

(i) A is r.e.

(v) $A = \emptyset$ or
$$A = \{(f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N}\}$$
for some primitive recursive functions
$$f_i : \mathbb{N} \to \mathbb{N} \ .$$

(vi) $A = \emptyset$ or
$$A = \{(f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N}\}$$
for some recursive functions
$$f_i : \mathbb{N} \to \mathbb{N} \ .$$

Theorem 8.5

Let $A \subseteq \mathbb{N}^n$. The following is equivalent:

(i) A is r.e.

(ii)
$$A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\}$$
for some primitive recursive predicate R.

(iii)
$$A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\}$$
for some recursive predicate R.

(iv)
$$A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\}$$
for some recursively enumerable predicate R.

Remark

- We can summarise Theorem 8.5 as follows:
There are 3 equivalent ways of defining that $A \subseteq \mathbb{N}^n$ is r.e.:
  - $A = \mathrm{dom}(f)$ for some partial recursive f;
  - $A = \emptyset$ or A is the image of primitive recursive/recursive functions $f_0, \ldots, f_{n-1}$;
  - $A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\}$ for some primitive recursive/recursive/r.e. R.

Remark, Case $n = 1$

- For $A \subseteq \mathbb{N}$ the following is equivalent:
  - A is r.e.
  - $A = \emptyset$ or
    $A = \mathrm{ran}(f)$ for some primitive recursive $f : \mathbb{N} \to \mathbb{N}$.
  - $A = \emptyset$ or $A = \mathrm{ran}(f)$ for some recursive $f : \mathbb{N} \to \mathbb{N}$.
- Therefore $A \subseteq \mathbb{N}$ is r.e., if
  - $A = \emptyset$
  - or there exists a (prim.-)rec. function f, which enumerates all its elements.
- This explains the name "recursively enumerable predicate".
  Skip Proof.

Proof Idea for Theorem 8.5:

- ((i) $\to$ (ii), Cont)
  - where

    $R(\vec{x}, y) \Leftrightarrow$ the TM for comp. $f(\vec{x})$ termin. after y steps

    R is primitive recursive.

Proof

Skip proof idea.

Proof Idea for Theorem 8.5:

- (i) $\to$ (ii):
  Assume A is r.e., $A = \mathrm{dom}(f)$, for f partial recursive.

$$
\begin{aligned}
A(\vec{x}) &\Leftrightarrow f(\vec{x}) \downarrow \\
&\Leftrightarrow \exists y.\text{the TM for computing } f(\vec{x}) \text{ terminates} \\
&\qquad \text{after } y \text{ steps} \\
&\Leftrightarrow \exists y.R(\vec{x}, y)
\end{aligned}
$$

Proof Ideas

- **(ii) $\to$ (v), special case $n = 1$:**
  Assume
  - $A = \{x \in \mathbb{N} \mid \exists y.R(x, y)\}$ where R is prim. rec.
  - $A \neq \emptyset$,
  - $y \in A$ fixed.

  Define $f : \mathbb{N} \to \mathbb{N}$ recursive,

$$
f(x) = \begin{cases} \pi_0(x), & \text{if } R(\pi_0(x), \pi_1(x)), \\ y & \text{otherwise.} \end{cases}
$$

  Then $A = \mathrm{ran}(f)$.

Proof Ideas

- **(v), (vi) $\rightarrow$ (i), special case $n = 1$:**
  Assume
  $$A = \mathsf{ran}(f) \ ,$$
  where f is (prim.-)recursive.
  Then
  $$A = \mathsf{dom}(g) \ ,$$
  where
  $$g(x) \simeq (\mu y.f(y) = x) \ .$$
  g is partial recursive.
- The full details will be omitted in the lecture.
  Skip Details

Proof of Theorem 8.5

- (i) $\rightarrow$ (ii) (Cont.)
  - Therefore

$$\begin{array}{lcl} A(\vec{x}) & \Leftrightarrow & \vec{x} \in \mathsf{dom}(f) \\ & \Leftrightarrow & \vec{x} \in \mathsf{dom}(\{e\}^n) \\ & \Leftrightarrow & \mathsf{U}(\mu y.\mathsf{T}_n(e, \vec{x}, y))\downarrow \\ & \overset{\text{U prim. rec., therefore total}}{\Leftrightarrow} & \mu y.\mathsf{T}_n(e, \vec{x}, y)\downarrow \\ & \Leftrightarrow & \exists y.\mathsf{T}_n(e, \vec{x}, y) \\ & \Leftrightarrow & \exists y.R(\vec{x}, y) \ . \end{array}$$

  where
  $$R(\vec{x}, y) \Leftrightarrow \mathsf{T}_n(e, \vec{x}, y) \ .$$

Proof of Theorem 8.5

- **(i) $\rightarrow$ (ii):**
  - (The actual predicate R we will take will be slightly differently from that in the proof idea – it is technically easier to prove the theorem this way.)
  - If A is r.e., then for some partial recursive function $f : \mathbb{N}^n \xrightarrow{\sim} \mathbb{N}$ we have
    $$A = \mathsf{dom}(f) \ .$$
  - Let $f = \{e\}^n$.
  - By Kleene's Normal Form Theorem there exist a primitive recursive function $\mathsf{U} : \mathbb{N} \rightarrow \mathbb{N}$ and a primitive recursive predicate $\mathsf{T}_n \subseteq \mathbb{N}^{n+1}$ s.t.
    $$\{e\}^n(\vec{x}) \simeq \mathsf{U}(\mu y.\mathsf{T}_n(e, \vec{x}, y)) \ .$$

Proof of Theorem 8.5

- (i) $\rightarrow$ (ii) (Cont.)
  - Now R is primitive recursive, and
    $$A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\} \ .$$

Proof of Theorem 8.5

- **(ii) $\rightarrow$ (iii)**: Trivial.
- **(iii) $\rightarrow$ (iv)**: By Lemma 8.3.

Proof of Theorem 8.5

- **(iv) $\rightarrow$ (ii)**:
  - Assume
    $$A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\} \enspace ,$$
    where R is r.e.
  - By "(i) $\rightarrow$ (ii)" there exists a primitive recursive predicate S s.t.
    $$R(\vec{x}, y) \Leftrightarrow \exists z.S(\vec{x}, y, z) \enspace .$$
  - Therefore
    $$\begin{aligned} A &= \{\vec{x} \mid \exists y.\exists z.S(\vec{x}, y, z)\} \\ &= \{\vec{x} \mid \exists y.S(\vec{x}, \pi_0(y), \pi_1(y))\} \\ &= \{\vec{x} \mid \exists y.R'(\vec{x}, y)\} \enspace , \end{aligned}$$

Proof of Theorem 8.5

- ((iv) $\rightarrow$ (ii), Cont.)
  - Here
    $$R'(\vec{x}, y) :\Leftrightarrow S(\vec{x}, \pi_0(y), \pi_1(y)) \text{ is primitive recursive.}$$

Proof of Theorem 8.5

- **(ii) $\rightarrow$ (v)**:
  - Assume A is not empty and R is primitive recursive s.t.
    $$A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\} \enspace .$$
  - Let $\vec{z} = z_0, \ldots, z_{n-1}$ be some fixed elements s.t. $A(\vec{z})$ holds.
  - Define for $i = 0, \ldots, n-1$
    $$f_i(x) := \begin{cases} \pi_i^{n+1}(x), & \text{if } R(\pi_0^{n+1}(x), \pi_1^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), \pi_n^{n+1}(x \\ z_i, & \text{otherwise.} \end{cases}$$
  - f_i are primitive recursive.

Proof of Theorem 8.5

- ((ii) → (v), Cont.)
  - We show

$$A = \{(f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N}\} \ .$$

Proof of Theorem 8.5

- ((ii) → (v), Cont.)
  - "⊇":
    Assume $x \in \mathbb{N}$, and show

$$A(f_0(x), \ldots, f_{n-1}(x)) \ .$$

    - If $R(\pi_0^{n+1}(x), \pi_1^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), \pi_n^{n+1}(x))$, then

$$\exists z.R(\pi_0^{n+1}(x), \pi_1^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), z) \ ,$$

      therefore

$$(\pi_0^{n+1}(x), \pi_1^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x)) \in A \ ,$$

      therefore

$$A(f_0(x), \ldots, f_{n-1}(x)) \ .$$

Proof of Theorem 8.5

- ((ii) → (v), Cont.)
  - ("⊇", Cont.):
    - If $(\mathbb{N}^k \setminus R)(\pi_0^{n+1}(x), \pi_1^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), \pi_n^{n+1}(x))$, then

$$f_i(x) = z_i \ ,$$

      therefore by $A(\vec{z})$

$$A(f_0(x), \ldots, f_{n-1}(x)) \ .$$

    So in both cases we get that

$$A(f_0(x), \ldots, f_{n-1}(x)) \ ,$$

    so

$$\{(f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N}\} \subseteq A \ .$$

Proof of Theorem 8.5

- ((ii) → (v), Cont.)
  - "⊆":
    - Assume

$$A(x_0, \ldots, x_{n-1}) \ ,$$

      and show

$$\exists z.(f_0(z) = x_0 \wedge \cdots \wedge f_{n-1}(z) = x_{n-1}) \ .$$

    - We have for some y

$$R(x_0, \ldots, x_{n-1}, y) \ .$$

    - Let

$$z = \pi^{n+1}(x_0, \ldots, x_{n-1}, y) \ .$$

Proof of Theorem 8.5

- ((ii) $\to$ (v), Cont.); ("$\subseteq$", Cont)
  - Then we have
$$x_i = \pi_i^{n+1}(z) \ , \ y = \pi_n^{n+1}(z) \ ,$$
    therefore
$$R(\pi_0^{n+1}(z), \pi_1^{n+1}(z), \dots, \pi_{n-1}^{n+1}(z), \pi_n^{n+1}(z)) \ ,$$
    therefore for $i = 0, \dots, n-1$
$$f_i(z) = \pi_i^{n+1}(z) = x_i \ ,$$

Proof of Theorem 8.5

- ((ii) $\to$ (v), Cont.); ("$\subseteq$", Cont)
  - therefore
$$(x_0, \dots, x_{n-1}) = (f_0(z), \dots, f_{n-1}(z)) \\ \in \{(f_0(x), \dots, f_{n-1}(x)) \mid x \in \mathbb{N}\} \ ,$$
  - and we have
$$A \subseteq \{(f_0(x), \dots, f_{n-1}(x)) \mid x \in \mathbb{N}\} \ .$$
  - Therefore we have shown
$$A = \{(f_0(x), \dots, f_{n-1}(x)) \mid x \in \mathbb{N}\} \ ,$$
    and the assertion follows.

Proof of Theorem 8.5

- **(v) $\to$ (vi)**: Trivial.
- **(vi) $\to$ (i)**:
  - If A is empty, then A is recursive, therefore r.e.
  - Assume
$$A = \{(f_0(x), \dots, f_{n-1}(x)) \mid x \in \mathbb{N}\} \ .$$
    for some recursive functions f_i.
  - Define
$$f : \mathbb{N}^n \xrightarrow{\sim} \mathbb{N} \ ,$$
    s.t.
$$f(x_0, \dots, x_{n-1}) :\simeq \mu x.(f_0(x) \simeq x_0 \wedge \cdots \wedge f_{n-1}(x) \simeq x_{n-1}$$

Proof of Theorem 8.5

- ((vi) $\to$ (i), Cont.)
  - f can be written as
$$\begin{aligned} f(x_0, \dots, x_{n-1}) :\simeq \mu x.(& ((f_0(x) \dot{-} x_0) + (x_0 \dot{-} f_0(x)))+ \\ & ((f_1(x) \dot{-} x_1) + (x_1 \dot{-} f_1(x)))+ \\ & \cdots + \\ & ((f_{n-1}(x) \dot{-} x_{n-1}) + (x_{n-1} \dot{-} f_{n-} \\ & \simeq 0) \ , \end{aligned}$$
    therefore f is partial recursive.

Proof of Theorem 8.5

- ((vi) → (i), Cont.)
  - Furthermore, we have

$$\begin{aligned} A(x_0,\ldots,x_{n-1}) &\Leftrightarrow \exists x \in \mathbb{N}.x_0 = f_0(x) \land \cdots \land x_{n-1} = f_{n-1}(x) \\ &\Leftrightarrow f(x_0,\ldots,x_{n-1})\downarrow \ , \end{aligned}$$

therefore

$$A = \text{dom}(f) \text{ is r.e.} \ .$$

Theorem 8.6

$A \subseteq \mathbb{N}^k$ is recursive iff both A and $\mathbb{N}^k \setminus A$ are r.e.

Proof idea:
"$\Rightarrow$" is easy.
For "$\Leftarrow$": Assume

$$\begin{aligned} A(\vec{x}) &\Leftrightarrow \exists y.R(\vec{x},y) \\ (\mathbb{N}^k \setminus A)(\vec{x}) &\Leftrightarrow \exists y.S(\vec{x},y) \end{aligned}$$

In order to decide A, search simultaneously for a y s.t. $R(\vec{x},y)$ and for a y s.t. $S(\vec{x},y)$ holds.
If we find a y s.t. $R(\vec{x},y)$ holds, then $A(\vec{x})$ holds.
If we find a y s.t. $S(\vec{x},y)$ holds, then $\neg A(\vec{x})$ holds

The details of the proof will be omitted in this lecture.

Jump over details

Proof of Theorem 8.6, "$\Rightarrow$"

- If A is recursive, then both A and $\mathbb{N}^k \setminus A$ are recursive, therefore as well r.e.

Proof of Theorem 8.6, "$\Leftarrow$"

- Assume A, $\mathbb{N}^k \setminus A$ are r.e.
- Then there exist primitive recursive predicates R and S s.t.

$$\begin{aligned} A &= \{\vec{x} \mid \exists y.R(\vec{x},y)\} \ , \\ \mathbb{N}^k \setminus A &= \{\vec{x} \mid \exists y.S(\vec{x},y)\} \ . \end{aligned}$$

Proof of Theorem 8.6, "$\Leftarrow$"

$A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\}$,
$\mathbb{N}^k \setminus A = \{\vec{x} \mid \exists y.S(\vec{x}, y)\}$.

- By
$$A \cup (\mathbb{N}^k \setminus A) = \mathbb{N}^k \ ,$$
it follows
$$\forall \vec{x}.((\exists y.R(\vec{x}, y)) \lor (\exists y.S(\vec{x}, y))) \ ,$$
therefore as well
$$\forall \vec{x}.\exists y.(R(\vec{x}, y) \lor S(\vec{x}, y)) \ . \qquad (*)$$

Proof of Theorem 8.6, "$\Leftarrow$"

$A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\}$, $\mathbb{N}^k \setminus A = \{\vec{x} \mid \exists y.S(\vec{x}, y)\}$,
$h(\vec{x}) := \mu y.(R(\vec{x}, y) \lor S(\vec{x}, y))$,
Show $A(\vec{x}) \Leftrightarrow R(\vec{x}, h(\vec{x}))$.

- If $A(\vec{x})$ then
$$\exists y.R(\vec{x}, y)$$
and
$$\vec{x} \notin (\mathbb{N}^k \setminus A) \ ,$$
therefore
$$\neg \exists y.S(\vec{x}, y) \ .$$
Therefore we have for the y found by $h(\vec{x})$ that $R(\vec{x}, y)$ holds, i.e.
$$R(\vec{x}, h(\vec{x})) \ .$$

Proof of Theorem 8.6, "$\Leftarrow$"

$A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\}$,
$\mathbb{N}^k \setminus A = \{\vec{x} \mid \exists y.S(\vec{x}, y)\}$,
$\forall \vec{x}.\exists y.(R(\vec{x}, y) \lor S(\vec{x}, y))$. $(*)$

- Define
$$h : \mathbb{N}^n \to \mathbb{N} \ , \ h(\vec{x}) := \mu y.(R(\vec{x}, y) \lor S(\vec{x}, y)) \ .$$
- h is partial recursive.
- By $(*)$ we have h is total, so h is recursive.
- We show
$$A(\vec{x}) \Leftrightarrow R(\vec{x}, h(\vec{x})) \ .$$

Proof of Theorem 8.6, "$\Leftarrow$"

$A = \{\vec{x} \mid \exists y.R(\vec{x}, y)\}$,
$\mathbb{N}^k \setminus A = \{\vec{x} \mid \exists y.S(\vec{x}, y)\}$,
$h(\vec{x}) := \mu y.(R(\vec{x}, y) \lor S(\vec{x}, y))$,
Show $A(\vec{x}) \Leftrightarrow R(\vec{x}, h(\vec{x}))$.

- On the other hand, if $R(\vec{x}, h(\vec{x}))$ holds then
$$\exists y.R(\vec{x}, y) \ ,$$
therefore
$$A(\vec{x}) \ .$$
Therefore
$$A = \{\vec{x} \mid R(\vec{x}, h(\vec{x}))\} \text{ is recursive.}$$

Theorem 8.7

Let $f : \mathbb{N}^n \xrightarrow{\sim} \mathbb{N}$.
Then

$$f \text{ is partial recursive} \Leftrightarrow \mathsf{G}_f \text{ is r.e. }.$$

Proof idea for "$\Leftarrow$":
Assume R primitive recursive s.t.

$$\mathsf{G}_f(\vec{x}, y) \Leftrightarrow \exists z.R(\vec{x}, y, z) \ .$$

In order to compute $f(\vec{x})$, search for a y s.t. $R(\vec{x}, \pi_0(y), \pi_1(y))$ holds.
$f(\vec{x})$ will be the first projection of this y.

The details of the proof will be omitted in this lecture.

Jump over details

Proof of Theorem 8.7, "$\Rightarrow$"

$f(\vec{x}) \simeq \mathsf{U}(\mu y.\mathsf{T}_n(\vec{x}, y))$.

- Therefore

$$\begin{aligned}(\vec{x}, y) \in \mathsf{G}_f &\Leftrightarrow (f(\vec{x}) \simeq y) \\ &\Leftrightarrow \exists z.(\mathsf{T}_n(\vec{x}, z) \wedge \\ &\qquad (\forall z' < z.\neg \mathsf{T}_n(\vec{x}, z')) \\ &\qquad \wedge \mathsf{U}(z) = y) \ ,\end{aligned}$$

- Therefore G_f is r.e.

Proof of Theorem 8.7, "$\Rightarrow$"

- Assume f is partial recursive.
- Then $f = \{e\}^n$ for some $e \in \mathbb{N}$.
- By Kleene's Normal Form Theorem we have

$$f(\vec{x}) \simeq \mathsf{U}(\mu y.\mathsf{T}_n(\vec{x}, y)) \ ,$$

  for some primitive recursive relation

$$\mathsf{T}_n \subseteq \mathbb{N}^{n+1}$$

  and some primitive recursive function

$$\mathsf{U} : \mathbb{N} \to \mathbb{N} \ .$$

Proof of Theorem 8.7, "$\Leftarrow$"

- If G_f is r.e., then there exists a primitive recursive predicate R s.t.

$$f(\vec{x}) \simeq y \Leftrightarrow (\vec{x}, y) \in \mathsf{G}_f \Leftrightarrow \exists z.R(\vec{x}, y, z) \ .$$

- Therefore for any z s.t. $R(\vec{x}, \pi_0(z), \pi_1(z))$ holds we have that

$$f(\vec{x}) \simeq \pi_0(z) \ .$$

- Therefore

$$f(\vec{x}) \simeq \pi_0(\mu u.R(\vec{x}, \pi_0(u), \pi_1(u))) \ ,$$

- f is partial recursive.

Lemma 8.8

The recursively enumerable sets are closed under:

- (a) **Union** (and therefore $\vee$):
  If $A, B \subseteq \mathbb{N}^n$ are r.e., so is $A \cup B$.
- (b) **Intersection** (and therefore $\wedge$):
  If $A, B \subseteq \mathbb{N}^n$ are r.e., so is $A \cap B$.
- (c) **Substitution by recursive functions:**
  If $A \subseteq \mathbb{N}^n$ is r.e., $f_i : \mathbb{N}^k \to \mathbb{N}$ are recursive for $i = 0, \ldots, n$, so is

$$C := \{\vec{y} \in \mathbb{N}^k \mid A(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y}))\} \ .$$

Proof of Lemma 8.8

- Let $A, B \subseteq \mathbb{N}^n$ be r.e.
- Then there exist primitive recursive relations R, S s.t.

$$A = \{\vec{x} \in \mathbb{N}^n \mid \exists y.R(\vec{x}, y)\} \ ,$$
$$B = \{\vec{x} \in \mathbb{N}^n \mid \exists y.S(\vec{x}, y)\} \ .$$

Lemma 8.8

- (d) **(Unbounded) existential quantification:**
  If $D \subseteq \mathbb{N}^{n+1}$ is r.e., so is

$$E := \{\vec{x} \in \mathbb{N}^n \mid \exists y.D(\vec{x}, y)\} \ .$$

- (e) **Bounded universal quantification:**
  If $D \subseteq \mathbb{N}^{n+1}$ is r.e., so is

$$F := \{(\vec{x}, z) \in \mathbb{N}^{n+1} \mid \forall y < z.D(\vec{x}, z)\} \ .$$

The details of the proof will be omitted in this lecture.

Jump over details

Proof of Lemma 8.8 (a), (b)

$A = \{\vec{x} \in \mathbb{N}^n \mid \exists y.R(\vec{x}, y)\}$,
$B = \{\vec{x} \in \mathbb{N}^n \mid \exists y.S(\vec{x}, y)\}$.

- One can easily see that

$$A \cup B = \{\vec{x} \in \mathbb{N}^n \mid \exists y.(R(\vec{x}, y) \vee S(\vec{x}, y))\} \ ,$$
$$A \cap B = \{\vec{x} \in \mathbb{N}^n \mid \exists y.(R(\vec{x}, \pi_0(y)) \wedge S(\vec{x}, \pi_1(y)))\} \ .$$

therefore $A \cup B$ and $A \cap B$ are r.e.

Proof of Lemma 8.8 (c)

$A = \{\vec{x} \in \mathbb{N}^n \mid \exists y.R(\vec{x}, y)\}$,
$B = \{\vec{x} \in \mathbb{N}^n \mid \exists y.S(\vec{x}, y)\}$.

- Assume $A \subseteq \mathbb{N}^n$ is r.e., $f_i : \mathbb{N}^k \to \mathbb{N}$ are recursive for $i = 0, \ldots, n$.
- Need to show that

$$C := \{(\vec{y} \in \mathbb{N}^k \mid A(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y})\} \ .$$

  is r.e.
- Follows by

$$\begin{aligned} C &= \{\vec{y} \mid A(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y}))\} \\ &= \{\vec{y} \mid \exists z.R(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y}), z)\} \text{ is r.e.} \end{aligned}$$

Proof of Lemma 8.8 (d), (e)

- **(d)** follows from Theorem 8.5.
- **(e):**
  - Assume T is a primitive recursive predicate s.t.

$$D = \{(\vec{x}, y) \in \mathbb{N}^{n+1} \mid \exists z.T(\vec{x}, y, z)\} \ .$$

  - Then we get

$$\begin{aligned} F &= \{(\vec{x}, y) \mid \forall y' < y.D(\vec{x}, y')\} \\ &= \{(\vec{x}, y) \mid \forall y' < y.\exists z.T(\vec{x}, y', z)\} \\ &= \{(\vec{x}, y) \mid \exists z.\forall y' < y.T(\vec{x}, y', (z)_{y'})\} \text{ is r.e.,} \end{aligned}$$

    where in the last line we used that

$$\{(\vec{x}, z) \mid \forall y' < y.T(\vec{x}, y', (z)_{y'})\} \text{ is primitive recursive} \ .$$

Lemma 8.9

The r.e. predicates are **not** closed under **complement**:

There exists an r.e. predicate $A \subseteq \mathbb{N}^n$ s.t. $\mathbb{N}^n \setminus A$ is not r.e.

Proof:

- Halt^n is r.e.
- $\mathbb{N}^n \setminus \mathsf{Halt}^n$ is not r.e.
  - Otherwise by Theorem 8.6 Halt^n would be recursive.
  - But by Lemma 8.3. (b) Halt^n is not recursive.