7. The Recursion Theorem

- Main result in this section:
  Kleene's Recursion Theorem.
  - Recursive functions are closed under a very general form of recursion.
- For the proof we will use the **S-m-n-theorem**.
  - Used in many proofs in computability theory.
  - However, both the S-m-n theorem and the proof of the Recursion theorem will be omitted this year.
    Jump to Kleene's Recursion Theorem.

The S-m-n Theorem

- Assume $f : \mathbb{N}^{m+n} \stackrel{\sim}{\rightarrow} \mathbb{N}$ partial recursive.
- Fix the first m arguments (say $\vec{l} := l_0, \ldots, l_{m-1}$).
- Then we obtain a partial recursive function

$$g : \mathbb{N}^n \stackrel{\sim}{\rightarrow} \mathbb{N} , \qquad g(\vec{x}) \simeq f(\vec{l}, \vec{x}) \ .$$

- The S-m-n theorem expresses that we can compute a Kleene index of g
  - i.e. an e' s.t. $g = \{e'\}^n$

  from a Kleene index of f and $\vec{l}$ **primitive recursively**.

The S-m-n Theorem

$f : \mathbb{N}^{m+n} \stackrel{\sim}{\rightarrow} \mathbb{N}$ partial rec.
$\vec{l} : \mathbb{N}^m$
$g : \mathbb{N}^n \stackrel{\sim}{\rightarrow} \mathbb{N}$ partial rec.
$g(\vec{x}) \simeq f(\vec{l}, \vec{x})$.

- So there exists a primitive recursive function S^m_n s.t.,
  - if $f = \{e\}^{m+n}$,
  - then $g = \{\mathsf{S}^m_n(e, \vec{l})\}^n$.
- So $\{\mathsf{S}^m_n(e, \vec{l})\}^n(\vec{x}) \simeq \{e\}^{m+n}(\vec{l}, \vec{x})$.

Notation

$\{\mathsf{S}^m_n(e, \vec{l})\}^n(\vec{x}) \simeq \{e\}^{m+n}(\vec{l}, \vec{x})$.

- Assume t is an expression depending on n variables $\vec{x}$, s.t. we can compute t from $\vec{x}$ partial recursively.
  Then $\lambda\vec{x}.t$ is any natural number e s.t. $\{e\}^n(\vec{x}) \simeq t$.
- Then we will have

$$\mathsf{S}^m_n(e, \vec{l}) = \lambda\vec{x}.\{e\}^{m+n}(\vec{l}, \vec{x}) \ .$$

Theorem 7.1 (S-m-n Theorem)

- Assume $m, n \in \mathbb{N}$.
- There exists a primitive recursive function

$$\mathsf{S}^m_n : \mathbb{N}^{m+1} \to \mathbb{N}$$

s.t. for all $\vec{l} \in \mathbb{N}^m, \vec{x} \in \mathbb{N}^n$

$$\{\mathsf{S}^m_n(e, \vec{l})\}^n(\vec{x}) \simeq \{e\}^{m+n}(\vec{l}, \vec{x}) \ .$$

Proof of S-m-n Theorem

- Let T be a TM encoded as e.
- A Turing machine T' corresponding to $\mathsf{S}^m_n(e, \vec{l})$ should be s.t.

$$\mathrm{T}'^{(n)}(\vec{x}) \simeq \mathrm{T}^{(n+m)}(\vec{l}, \vec{x}) \ .$$

Proof of S-m-n Theorem

T is TM for e.
Want to define T' s.t. $\mathrm{T}'^{(n)}(\vec{x}) \simeq \mathrm{T}^{(n+m)}(\vec{l}, \vec{x})$
T' can be defined as follows:

1. The initial configuration is:
   - $\vec{x}$ written on the tape,
   - head pointing to the left most bit:

$\cdots$	$\sqcup$	$\sqcup$	$\mathsf{bin}(x_0)$	$\sqcup$	$\cdots$	$\sqcup$	$\mathsf{bin}(x_{n-1})$	$\sqcup$	$\sqcup$	$\cdots$
			$\uparrow$							

Proof of S-m-n Theorem

T is TM for e.
Want to define T' s.t. $\mathrm{T}'^{(n)}(\vec{x}) \simeq \mathrm{T}^{(n+m)}(\vec{l}, \vec{x})$
Initial configuration:

$\cdots$	$\sqcup$	$\sqcup$	$\mathsf{bin}(x_0)$	$\sqcup$	$\cdots$	$\sqcup$	$\mathsf{bin}(x_{n-1})$	$\sqcup$	$\sqcup$	$\cdots$
			$\uparrow$							

2. - T' writes first binary representation of $\vec{l} = l_0, \ldots, l_{n-1}$ in front of this.
   - terminates this step with the head pointing to the most significant bit of $\mathsf{bin}(l_0)$.

   So configuration after this step is:

$\mathsf{bin}(l_0)$	$\sqcup$	$\cdots$	$\sqcup$	$\mathsf{bin}(l_{m-1})$	$\sqcup$	$\mathsf{bin}(x_0)$	$\sqcup$	$\cdots$	$\sqcup$	bi
$\uparrow$										

Proof of S-m-n Theorem

T is TM for e.
Want to define T' s.t. $\mathrm{T}'^{(n)}(\vec{x}) \simeq \mathrm{T}^{(n+m)}(\vec{l}, \vec{x})$.
Configuration after first step:

$\mathsf{bin}(l_0)$	␣␣	$\cdots$	␣␣	$\mathsf{bin}(l_{m-1})$	␣␣	$\mathsf{bin}(x_0)$	␣␣	$\cdots$	␣␣	$\mathsf{bin}(x_{n-1})$

$\uparrow$

- Then T' runs T, starting in this configuration.
  It terminates, if T terminates.
  The result is
  $$\simeq \mathrm{T}^{(m+n)}(\vec{l}, \vec{x}) \ ,$$
  and we get therefore
  $$\mathrm{T}'^{(n)}(\vec{x}) \simeq \mathrm{T}^{(m+n)}(\vec{l}, \vec{x})$$
  as desired.

Proof of the S-m-n Theorem

- A code for T' can be obtained from a code for T and from $\vec{l}$ as follows:
  - One takes a Turing machine T'', which writes the binary representations of
    $$\vec{l} = l_0, \ldots, l_{m-1}$$
    in front of its initial position (separated by a blank and with a blank at the end), and terminates at the left most bit.
  - It's a straightforward exercise to write a code for the instructions of such a Turing machine, depending on $\vec{l}$, and show that the function defining it is primitive recursive.

Proof of the S-m-n Theorem

T is TM for e.
T' is a TM s.t. $\mathrm{T}'^{(n)}(\vec{x}) \simeq \mathrm{T}^{(n+m)}(\vec{l}, \vec{x})$

- From a code for T one can now obtain a code for T' in a primitive recursive way.
- S^m_n is the corresponding function.
- The details will not be given in the lecture
  Jump to Kleene's Recursion Theorem

Proof of the S-m-n Theorem

- Assume, the terminating state of T'' has Gödel number (i.e. code) s, and that all other states have Gödel numbers $< s$.
- Then one appends to the instructions of T'' the instructions of T, but with the states shifted, so that the new initial state of T is the final state s of T'' (i.e. we add s to all the Gödel numbers of states occurring in T).
- This can be done as well primitive recursively.

Proof of the S-m-n Theorem

- So a code for T'' can be defined primitive recursively depending on a code e for T and $\vec{l}$, and S^m_n is the primitive recursive function computing this. With this function it follows now that, if e is a code for a TM, then

$$\{\mathsf{S}^m_n(e,\vec{l})\}^n(\vec{x}) \simeq \{e\}^{n+m}(\vec{l},\vec{x}) \ .$$

  This equation holds, even if e is not a code for a TM: In this case $\{e\}^{m+n}$ interprets e as if it were the code for a valid TM T

Proof of the S-m-n Theorem

- $e' := \mathsf{S}^m_n(e,\vec{l})$ will have the same deficiencies as e, but when applying the Kleene-brackets, it will be interpreted as a TM T' obtained from e' in the same way as we obtained T from e, and therefore

$$\{e'\}^n(\vec{x}) \simeq \mathrm{T}'^{(n)}(\vec{x}) \simeq \mathrm{T}^{(n+m)}(\vec{l},\vec{x}) \simeq \{e\}^{n+m}(\vec{l},\vec{x}) \ .$$

  So we obtain the desired result in this case as well.

Proof of the S-m-n Theorem

- (A code for such a valid TM is obtained by
  - deleting any instructions $\mathrm{encode}(q,a,q',a',D)$ in e s.t. there exists an instruction $\mathrm{encode}(q,a,q'',a'',D')$ occurring before it in the sequence e,
  - and by replacing all directions > 1 by $\ulcorner \mathsf{R} \urcorner = 1$.)

Kleene's Recursion Theorem

- Assume $f : \mathbb{N}^{n+1} \xrightarrow{\sim} \mathbb{N}$ partial recursive.
- Then there exists an $e \in \mathbb{N}$ s.t.

$$\{e\}^n(\vec{x}) \simeq f(e,\vec{x}) \ .$$

  (Here $\vec{x} = x_0, \ldots, x_{n-1}$).

Example 1

Kleene's Rec. Theorem: $\exists e.\forall \vec{x}.\{e\}^n(\vec{x}) \simeq f(e,\vec{x})$.

- There exists an e s.t.

$$\{e\}(x) \simeq e + 1 \ .$$

For showing this take in the Recursion Theorem $f(e,n) := e + 1$.
Then

$$\{e\}(x) \simeq f(e,x) \simeq e + 1 \ .$$

Example 2

- The function computing the **Fibonacci-numbers** fib is recursive.
  - (This is a weaker result than what we obtained above –
  - above we showed that it is even prim. rec.)

Remark

Kleene's Rec. Theorem: $\exists e.\forall \vec{x}.\{e\}^n(\vec{x}) \simeq f(e,\vec{x})$.

- Applications as Example 1 are usually not very useful.
- Usually, when using the Rec. Theorem, one
  - doesn't use the index e directly,
  - but only the application of $\{e\}$ to arguments.

Fibonacci Numbers

Remember the defining equations for fib:

$$\begin{aligned} \mathsf{fib}(0) &= \mathsf{fib}(1) = 1 \ , \\ \mathsf{fib}(n+2) &= \mathsf{fib}(n) + \mathsf{fib}(n+1) \ . \end{aligned}$$

From these equations we obtain

$$\mathsf{fib}(n) = \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ \mathsf{fib}(n \dot{-} 2) + \mathsf{fib}(n \dot{-} 1), & \text{otherwise.} \end{cases}$$

We show that there exists a recursive function $g : \mathbb{N} \to \mathbb{N}$, s.t.

$$g(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ g(n \dot{-} 2) + g(n \dot{-} 1), & \text{otherwise.} \end{cases}$$

Fibonacci Numbers

Show: Exists g rec.

s.t. $g(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ g(n \dot{-} 2) + g(n \dot{-} 1), & \text{otherwise.} \end{cases}$

Shown as follows: Define a recursive $f : \mathbb{N}^2 \to \mathbb{N}$ s.t.

$$f(e,n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ \{e\}(n \dot{-} 2) + \{e\}(n \dot{-} 1), & \text{otherwise.} \end{cases}$$

Now let e be s.t.

$$\{e\}(n) \simeq f(e,n) \ .$$

Then e fulfils the equations

$$\{e\}(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ \{e\}(n \dot{-} 2) + \{e\}(n \dot{-} 1), & \text{otherwise.} \end{cases}$$

General Applic. of Rec. Theorem

- Similarly, one can introduce arbitrary partial recursive functions g, where
  - $g(\vec{n})$ refers to arbitrary other values $g(\vec{m})$.
- So, instead of arguing as before that fib is partial recursive, it suffices to say the following
  - By the recursion theorem, there exists a partial recursive function $\mathsf{fib} : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$, s.t.

$$\mathsf{fib}(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ \mathsf{fib}(n \dot{-} 2) + \mathsf{fib}(n \dot{-} 1), & \text{otherwise.} \end{cases}$$

  - We can prove by induction on n that $\forall n : \mathbb{N}.\mathsf{fib}(n)\downarrow$ holds.
  - Therefore fib is total and therefore recursive.

Fibonacci Numbers

$$\{e\}(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ \{e\}(n \dot{-} 2) + \{e\}(n \dot{-} 1), & \text{otherwise.} \end{cases}$$

Let $g = \{e\}$.
Then we get

$$g(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ g(n \dot{-} 2) + g(n \dot{-} 1), & \text{otherwise.} \end{cases}$$

These are the defining equations for fib.
One can show by induction on n that $g(n) = \mathsf{fib}(n)$ for all $n \in \mathbb{N}$.
Therefore fib is recursive.

General Applic. of Rec. Theorem

- This use of the the recursion theorem corresponds to the recursive definition of functions in programming.
- E.g. in Java one defines

```
public static int fib(int n){
  if (n == 0 || n == 1){
    return 1;}
  else{
    return fib(n-1) + fib(n-2);
  }
};
```

Example 3

As in general programming, recursively defined functions need not be total:

- There exists a partial recursive function $g : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ s.t.

$$g(x) \simeq g(x) + 1 \ .$$

- We get $g(x)\uparrow$.
- The definition of g corresponds to the following Java definition:

```
public static int g(int n){
  return g(n) + 1;
};
```

- When executing $g(x)$, Java loops.

Ackermann Function

- The Ackermann function is recursive: Remember the defining equations:

$$\begin{aligned} \mathsf{Ack}(0,y) &= y+1 \ , \\ \mathsf{Ack}(x+1,0) &= \mathsf{Ack}(x,1) \ , \\ \mathsf{Ack}(x+1,y+1) &= \mathsf{Ack}(x,\mathsf{Ack}(x+1,y)) \ . \end{aligned}$$

- From this we obtain

$$\mathsf{Ack}(x,y) = \begin{cases} y+1, & \text{if } x=0, \\ \mathsf{Ack}(x \dot{-} 1, 1), & \text{if } x>0 \text{ and } y= \\ \mathsf{Ack}(x \dot{-} 1, \mathsf{Ack}(x, y \dot{-} 1)), & \text{otherwise.} \end{cases}$$

Example 4

- There exists a partial recursive function $g : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ s.t.

$$g(x) \simeq g(x+1) + 1 \ .$$

  Note that that's a "black hole recursion", which is not solvable by a total function.
- It is solved by $g(x)\uparrow$.
- Note that a recursion equation for a function f cannot always be solved by setting $f(x)\uparrow$.
  - E.g. the recursion equation for fib can't be solved by setting $\mathsf{fib}(n)\uparrow$.

Ackermann Function

$$\mathsf{Ack}(x,y) = \begin{cases} y+1, & \text{if } x=0, \\ \mathsf{Ack}(x \dot{-} 1, 1), & \text{if } x>0 \text{ and } y=0, \\ \mathsf{Ack}(x \dot{-} 1, \mathsf{Ack}(x, y \dot{-} 1)), & \text{otherwise.} \end{cases}$$

- Define g partial recursive s.t.

$$g(x,y) \simeq \begin{cases} y+1, & \text{if } x=0, \\ g(x \dot{-} 1, 1), & \text{if } x>0 \wedge y=0, \\ g(x \dot{-} 1, g(x, y \dot{-} 1)), & \text{if } x>0 \wedge y>0. \end{cases}$$

- g fulfils the defining equations of Ack.
- Proof that $g(x,y) \simeq \mathsf{Ack}(x,y)$ follows by main induction on x, side-induction on y. The details will not be given in the lecture. **Jump over remaining slides**.

Proof of Correctness of Ack

- We show by induction on x that $g(x,y)$ is defined and equal to $\mathrm{Ack}(x,y)$ for all $x,y \in \mathbb{N}$:
  - Base case $x = 0$.

$$g(0,y) = y+1 = \mathrm{Ack}(0,y) \ .$$

  - Induction Step $x \to x+1$. Assume

$$g(x,y) = \mathrm{Ack}(x,y) \ .$$

  We show

$$g(x+1,y) = \mathrm{Ack}(x+1,y)$$

  by side-induction on y:

Proof of Correctness of Ack

Show $g(x+1,y) = \mathrm{Ack}(x+1,y)$

- Base case $y = 0$:

$$g(x+1,0) \simeq g(x,1) \overset{\text{Main-IH}}{=} \mathrm{Ack}(x,1) = \mathrm{Ack}(x+1,0) \ .$$

- Induction Step $y \to y+1$:

$$\begin{array}{rcl} g(x+1,y+1) & \simeq & g(x, g(x+1,y)) \\ & \overset{\text{Main-IH}}{\simeq} & g(x, \mathrm{Ack}(x+1,y)) \\ & \overset{\text{Side-IH}}{\simeq} & \mathrm{Ack}(x, \mathrm{Ack}(x+1,y)) \\ & = & \mathrm{Ack}(x+1,y+1) \ . \end{array}$$

Jump over remaining slides (Proof of the Recursion Theorem)

Idea of Proof of the Rec. Theorem

Assume

$$f : \mathbb{N}^{n+1} \xrightarrow{\sim} \mathbb{N} \ .$$

We have to find an e s.t.

$$\forall \vec{x} \in \mathbb{N}.\{e\}^n(\vec{x}) \simeq f(e,\vec{x}) \ .$$

- We set $e = \lambda\!\!\lambda \vec{x}.\{e_1\}^{n+1}(e_1,\vec{x})$ for some e_1 to be determined.
- Then the left and right hand side of the equation of the recursion theorem reads

$$\begin{array}{rcl} \{e\}^n(\vec{x}) & \simeq & \{\lambda\!\!\lambda \vec{x}.\{e_1\}^{n+1}(e_1,\vec{x})\}^n(\vec{x}) \\ & \simeq & \{e_1\}^{n+1}(e_1,\vec{x}) \\ f(e,\vec{x}) & \simeq & f(\lambda\!\!\lambda \vec{x}.\{e_1\}^{n+1}(e_1,\vec{x}),\vec{x}) \end{array}$$

Idea Proof of Rec. Theorem

We need to satisfy $\forall \vec{x} \in \mathbb{N}.\{e\}^n(\vec{x}) \simeq f(e,\vec{x})$.
Let $e = \lambda\!\!\lambda \vec{x}.\{e_1\}^{n+1}(e_1,\vec{x})$.

$$\begin{array}{rcl} \{e\}^n(\vec{x}) & \simeq & \{e_1\}^{n+1}(e_1,\vec{x}) \ , \\ f(e,\vec{x}) & \simeq & f(\lambda\!\!\lambda \vec{x}.\{e_1\}^{n+1}(e_1,\vec{x}),\vec{x}) \ . \end{array}$$

- So e_1 needs to fulfill the following equation:

$$\begin{array}{rcl} \{e_1\}^{n+1}(e_1,\vec{x}) & \simeq & \{e\}^n(\vec{x}) \\ & \overset{!}{\simeq} & f(e,\vec{x}) \\ & \simeq & f(\lambda\!\!\lambda \vec{x}.\{e_1\}^{n+1}(e_1,\vec{x}),\vec{x}) \end{array}$$

- This can be fulfilled if we define e_1 s.t.

$$\{e_1\}^{n+1}(e_2,\vec{x}) \simeq f(\lambda\!\!\lambda \vec{x}.\{e_2\}^{n+1}(e_2,\vec{x}),\vec{x})$$

Idea of Proof of Rec. Theorem

$\{e_1\}^{n+1}(e_2, \vec{x}) \simeq f(\lambda\vec{x}.\{e_2\}^{n+1}(e_2, \vec{x}), \vec{x})$.

- By the S-m-n Theorem we can obtain this if we have e_1 s.t.

$$\{e_1\}^{n+1}(e_2, \vec{x}) \simeq f(\mathsf{S}^1_n(e_2, e_2), \vec{x})$$

- There exists a partial recursive function $g : \mathbb{N}^n + 1 \stackrel{\sim}{\to} \mathbb{N}$, s.t.

$$g(e_2, \vec{x}) \simeq f(\mathsf{S}^1_n(e_2, e_2), \vec{x})$$

- If e_1 is an index for g we obtain the desired equation.

$$\{e_1\}^{n+1}(e_2, \vec{x}) \simeq f(\mathsf{S}^1_n(e_2, e_2), \vec{x})$$

Complete Proof of Rec. Theorem

Let e_1 be s.t.

$$\{e_1\}^{n+1}(y, \vec{x}) \simeq f(\mathsf{S}^1_n(y, y), \vec{x}) \quad .$$

Let $e := \mathsf{S}^1_n(e_1, e_1)$.
Then we have

$$\begin{array}{lcl}
\{e\}^n(\vec{x}) & \stackrel{e = \mathsf{S}^1_n(e_1, e_1)}{\simeq} & \{\mathsf{S}^1_n(e_1, e_1)\}^n(\vec{x}) \\
 & \stackrel{\text{S-m-n theorem}}{\simeq} & \{e_1\}^{n+1}(e_1, \vec{x}) \\
 & \stackrel{\text{Def of } e_1}{\simeq} & f(\mathsf{S}^1_n(e_1, e_1), \vec{x}) \\
 & \stackrel{e = \mathsf{S}^1_n(e_1, e_1)}{\simeq} & f(e, \vec{x}) \quad .
\end{array}$$