A BUCHOLZ DERIVATION SYSTEM FOR THE ORDINAL ANALYSIS OF $\mathbf{KP} + \Pi_3$-REFLECTION

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Abstract. In this paper we introduce a notation system for the infinitary derivations occurring in the ordinal analysis of $\mathbf{KP} + \Pi_3$-Reflection due to Michael Rathjen. This allows a finitary ordinal analysis of $\mathbf{KP}+\Pi_3$-Reflection. The method used is an extension of techniques developed by Wilfried Buchholz, namely operator controlled notation systems for RS^∞-derivations. Similarly to Buchholz we obtain a characterisation of the provably recursive functions of $\mathbf{KP}+\Pi_3$-Reflection as $<$-recursive functions where $<$ is the ordering on Rathjen's ordinal notation system $\mathcal{T}(K)$. Further we show a conservation result for Π^0_2-sentences.

§1. Introduction. Ordinal analysis uses cut-elimination techniques for proof theoretic investigations. The termination of the cut-elimination process is guaranteed by assigning decreasing ordinals to the proofs emerging in the process. Gerhard Gentzen was the first to form a relationship between an ordinal ε_0 and a foundational mathematical theory (nowadays denoted Peano Arithmetic **PA**) in this way. Kurt Schütte [25] showed that cut-elimination can be radically simplified by moving to an infinitary proof calculus which allows the embedding of **PA**. This is made possible by replacing the generalisation rule by the infinitary ω-rule

$$\frac{\cdots A(\bar{n}) \cdots (n \in \omega)}{\forall x A(x)}$$

and by only working with sentences (formulas without free variables). The ordinal assignment for this infinitary derivations is now given by the length of the derivation. This work clarified the relationship between ε_0 and **PA**. Since this time infinitary methods have been successfully applied for the analysis of numerous other theories (e.g. [13, 9, 20, 22, 23] to name just a few). However as pointed out by Wilfried Buchholz [5] something is lost by passing from finite to infinite derivations. So Gentzen's method gives us bounds for the provably recursive functions, conservation results or the unprovability of primitive recursive wellfoundedness PRWO. To recapture these results when working with infinitary derivations we need the (primitive) recursion theorem. However citing Buchholz again "this requires a lot of cumbersome and boring coding machinery which on

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the other side is not completely trivial, and it seems to me that all presentations of this subject in the existing literature are more ore less unsatisfactory". We agree to the latter as well.

In this paper we continue work of Buchholz [5, 6, 7, 8] and Tupailo [32]. We define a finitary notation system for the infinitary derivations occurring in the ordinal analysis of $\mathbf{KP} + \Pi_3$-Reflection due to Michael Rathjen [20, 22]. This gives a finitary ordinal analysis of the axiom system $\mathbf{KP} + \Pi_3$-Reflection. As an application of our notation system we give a characterisation of the provably recursive functions of the theory. Further we prove a conservation result. We think that the methods used here may brought forward in an relatively schematic way to a finitary treatment of the infinitary ordinal analysis of Π^1_2-$\mathbf{CA}$ due to Michael Rathjen. There is as well the hope that this work may contribute in comparing the work of Michael Rathjen [22, 23, 24, 19] and Toshiyasu Arai [1, 2].

The paper is organised as follows: The first three sections are completely devoted to the citation of definitions and results. In section 2 we recall definitions and properties of Rathjens ordinal notation system $\mathcal{T}(K)$ and in section 3 the definition of the language of ramified set theory. In section 4 we remind the reader on Buchholz [8] notions of inferences, derivations and proof systems. In section 5 we transfer Michael Rathjens proof system $RS(\mathcal{K})$ into our new framework. In the following three sections we proceed again as in Buchholz. Unfortunately we can not simply cite the definitions and results but have to do some minor changes. In section 6 we adapt Buchholz definition of what it means to be a notation system to our purpose. A finitary notation system for infinitary derivations consists essentially of notations for some infinitary derivations and maps which assign to these notations

1. the last inference of the denoted infinitary derivation,
2. ordinals of the infinitary derivation to measure height, cut rank, etc.,
3. notations for the sub derivations.

In section 7 we give a notation system for embedding the axioms of $\mathbf{KP} + \Pi_3$-Reflection and in section 8 we define notations for derivations for the logically valid formulas. In section 9 the actual work starts. We specify the inference rules for the cut elimination procedure. First we work with the closure of the notations of the sections 7, 8 under this rules. In the following section 10 we assign ordinals $\mathsf{o}(h)$, $\mathsf{deg}(h)$, $\mathsf{ref}(h)$ to the notations. In section 12 we assign to every notation h a rule $\mathsf{tp}(h)$ which correspond to the last rule in the denoted infinitary derivation d. Further we assign to every index i of the premises of $\mathsf{tp}(h)$ a notation $h[i]$ for the corresponding sub derivation of d. The essential point here is that all this can be done in a primitive recursive way. We don't need transfinite recursion. In section 13 we conclude the definition of our notation system by restricting the use of the inference rules given in section 9. In section 14 we prove our main result namely that we have gained a notation system in the sense of section 6. A closer look to the proof shows that we can prove the result in a very weak theory namely Primitive Recursive Arithmetic. This is an important condition to prove the above mentioned applications in the last two sections: a characterisation of the provably recursive functions and a conservation result for $\mathbf{KP} + \Pi_3$-Reflection. Some knowledge of the work of Buchholz [5, 6, 7, 8] and

Rathjen [20, 22] is helpful to understand the paper. We recommend especially the reading of [8] and [20, 22].

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§2. The ordinal notation system $\mathcal{T}(K)$. In this section we recall the definitions and properties used in [20] to define a primitive recursive set $\mathcal{T}(K)$ together with a well ordering $<$ on $\mathcal{T}(K)$ which is primitive recursive as well. We assume in this section the existence of a weakly compact cardinal $\mathcal{K}$. Weakly compact cardinals are Π^1_1-indescribable [14]. This property of $\mathcal{K}$ is used only once. All following theorems are quoted from [20]. We are only interested in the properties stated. Therefore we refrain from giving the exact definitions and proofs. The reader may find them in [10, 16]. The properties stated here are used heavily in the proof of Theorem 12.4. We start by fixing some basic notions. Our main references for this section are [26, 18, 20]. We use the notations $On, Card, Lim$ for the classes of ordinals, infinite cardinals and limit ordinals respectively. Small Greek letters are reserved for ordinals with one exception: φ which is used for the Veblen function. The ordering on On is denoted by $<$. With $]\alpha,\beta[$, $[\alpha,\beta[$, $[\alpha,\beta]$ we denote open, half open and closed intervals of ordinals. For a class M of ordinals we write ord_M for its enumeration function. For the class *card* we use as well the standard notation $\alpha \mapsto \Omega_\alpha$. We have

$$\mathsf{dom}(ord_M) = On \Leftrightarrow M \text{ unbounded in } On.$$

A function $f : M \to On$ where $M \subseteq On$ is called continuous if f is continuous with respect to the order topology on On. A strong monotone continuous function f with $\mathsf{dom}(f) = On$ ($\mathsf{dom}(f) = \rho, \rho$ regular cardinal) is called a normal function (on ρ). This is equivalent to being the enumeration function of a closed and unbounded class of ordinals in On (in ρ). Such classes are called clubs (from closed and unbounded). A class M of ordinals is called stationary in On (in ρ) if M has a non empty intersection with every club in On (in ρ). This is equivalent to the requirement that every normal function (on ρ) has to have a fix point in M. An ordinal ρ is called regular if the cofinality of ρ is ρ. This is equivalent to: every subset of ρ with cardinality smaller than ρ is bounded in ρ. The fix points of a normal function on $\rho > \omega$ form a club. We denote the class of regular cardinals above ω by Reg. We use the small Greek letters π, τ, κ (possibly with indices) for regular cardinals $< \mathcal{K}$. An ordinal $\gamma > 0$ is called an (additive) principal if it is closed under ordinal addition, i.e. $\forall \alpha, \beta < \gamma . \alpha + \beta < \gamma$. We denote the class of additive principals by H (from German "Hauptzahlen"). The enumeration function of H is given by $\alpha \mapsto \omega^\alpha$. We recall the following basic facts and definitions:

Proposition 2.1. *For $\alpha \notin H \cup \{0\}$ exist unique $\alpha_1, ...\alpha_n \in H$ with $\alpha_n \leq ... \leq \alpha_1 < \alpha$ and $\alpha = \alpha_1 + ... + \alpha_n$.*

Definition 2.2 (Cantor normal form).
$\alpha =_{\mathsf{NF}} \alpha_1 + ... + \alpha_n :\Leftrightarrow \alpha = \alpha_1 + ... + \alpha_n$, $\alpha_1, ...\alpha_n \in H$ and $\alpha_n \leq ... \leq \alpha_1 < \alpha$.

Definition 2.3 (Veblen Hierarchy). By induction on α

$$\varphi_\alpha := ord_{\{\xi \in H : \forall \eta < \alpha . \varphi_\eta(\xi) = \xi\}}.$$

Notations. We write $\varphi\alpha\beta$ for $\varphi_\alpha(\beta)$.

φ_α is a normal function on On for $\alpha \in On$.

Proposition 2.4. (a) $\varphi\alpha\beta \in H$
(b) $\xi < \alpha \Rightarrow \varphi\xi(\varphi\alpha\beta) = \varphi\alpha\beta$
(b) $\beta < \gamma \Rightarrow \varphi\alpha\beta < \varphi\alpha\gamma$
(c) $\alpha < \beta \Rightarrow \varphi\alpha 0 < \varphi\beta 0$
(d) $\alpha, \beta \leq \varphi\alpha\beta$.

Theorem 2.5. $\varphi\alpha_1\beta_1 = \varphi\alpha_2\beta_2$ *iff*

(i) $\alpha_1 < \alpha_2$ *and* $\beta_1 = \varphi\alpha_2\beta_2$ *or*
(ii) $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$ *or*
(iii) $\alpha_2 < \alpha_1$ *and* $\beta_2 = \varphi\alpha_1\beta_1$.

Theorem 2.6. $\varphi\alpha_1\beta_1 < \varphi\alpha_2\beta_2$ *iff*

(i) $\alpha_1 < \alpha_2$ *and* $\beta_1 < \varphi\alpha_2\beta_2$ *or*
(ii) $\alpha_1 = \alpha_2$ *and* $\beta_1 < \beta_2$ *or*
(iii) $\alpha_2 < \alpha_1$ *and* $\varphi\alpha_1\beta_1 < \beta_2$.

Theorem 2.7. $\forall\gamma \in H.\exists!\alpha, \beta.\gamma = \varphi\alpha\beta$ *and* $\beta < \gamma$.

Definition 2.8. An ordinal γ is called strongly critical if $\varphi\gamma 0 = \gamma$. We denote the class of strongly critical ordinals by S.

An equivalent definition is that $\gamma \neq 0$ is closed under the φ-function i.e. $\forall\alpha, \beta < \gamma.\varphi\alpha\beta < \gamma$.

Proposition 2.9. $\forall\gamma \in H \setminus S.\exists!\alpha, \beta < \gamma.\gamma = \varphi\alpha\beta$.

Definition 2.10. $\gamma =_{\mathsf{NF}} \varphi\alpha\beta :\Leftrightarrow \gamma = \varphi\alpha\beta$ and $\alpha, \beta < \gamma$

Definition 2.11. We define a finite set of ordinals $S(\gamma)$ by:

1. $S(\gamma) := \{\gamma\}$ for $\gamma \in S \cup \{0\}$,
2. $S(\gamma) := \{\gamma_1, ..., \gamma_n\}$ for $\gamma =_{\mathsf{NF}} \gamma_1 + ... + \gamma_n \notin H$,
3. $S(\gamma) := \{\alpha, \beta\}$ for $\gamma =_{\mathsf{NF}} \varphi\alpha\beta \in H \setminus S$.

Proposition 2.12. $\gamma \notin S \cup \{0\}$ *iff* $S(\gamma) < \gamma$.

For proofs of the above see e.g. [26, 18]. We have:

$$Reg \cup \{\omega\} \subseteq Card \subseteq S \subseteq H \subseteq Lim \subseteq On.$$

DEFINITION 2.13. We define sets of ordinals $C^n(\alpha,\beta), C(\alpha,\beta), M^\alpha$ and ordinals $\Xi(\alpha), \Psi^\xi_\pi(\alpha)$ by main recursion on α with side induction on n as follows:

$$\begin{aligned}
C^0(\alpha,\beta) &:= \beta \cup \{0,\mathcal{K}\}\\
C^{n+1}(\alpha,\beta) &:= C^n(\alpha,\beta) \cup \{\xi+\eta : \xi,\eta \in C^n(\alpha,\beta)\}\\
&\cup\{\varphi\xi\eta : \xi,\eta \in C^n(\alpha,\beta)\}\\
&\cup\{\Omega_\xi : \xi < \mathcal{K} \wedge \xi \in C^n(\alpha,\beta)\}\\
&\cup\{\Xi(\xi) : \xi < \alpha \wedge \xi \in C^n(\alpha,\beta)\}\\
&\cup\{\Psi^\xi_\pi(\delta) : \xi \leq \delta < \alpha \wedge \xi,\pi,\delta \in C^n(\alpha,\beta)\}
\end{aligned}$$

$$C(\alpha,\beta) := \bigcup_{n<\omega} C^n(\alpha,\beta)$$

$$\begin{aligned}
M^0 &:= \mathcal{K} \cap Lim\\
M^\alpha &:= \{\pi < \mathcal{K} : C(\alpha,\pi) \cap \mathcal{K} = \pi \wedge\\
&\qquad \forall \xi \in C(\alpha,\pi) \cap \alpha . M^\xi \text{ stationary in } \pi \wedge\\
&\qquad \alpha \in C(\alpha,\pi)\}\\
&\text{for } \alpha > 0
\end{aligned}$$

$$\begin{aligned}
\Xi(\alpha) &:= \min(M^\alpha \cup \{\mathcal{K}\})\\
\Psi^\xi_\pi(\alpha) &:= \min(\{\rho \in M^\xi \cap \pi : C(\alpha,\rho) \cap \pi = \rho \wedge \pi, \alpha \in C(\alpha,\rho)\} \cup \{\pi\})\\
&\text{for } \xi \leq \alpha
\end{aligned}$$

The set $C(\alpha,\beta)$ is the closure of $\beta \cup \{0,\mathcal{K}\}$ under the (partial) functions $+, \varphi, \xi \mapsto \Omega_\xi, \Xi, \Psi$. We call an ordinal α Mahlo on X for $X \subseteq On$ if for every function $f : \alpha \to \alpha$ exists a f closed $\beta \in (X \cap \alpha) \setminus \{0\}$ i.e. $\eta < \beta \Rightarrow f(\eta) < \beta$. Let $M(X) := \{\alpha \in X : \alpha \text{ is Mahlo on } X\}$. The class M^1 is the class of Mahlo cardinals $M(Reg)$. For $\xi < \mathcal{K}$ is M^ξ the image of Reg under the ξ-times iterated operator M. If we reach $\mathcal{K}$ we diagonalise the first time i.e. $M^\mathcal{K} = \{\alpha : \alpha \in M^\alpha\}$. The class $M^{\mathcal{K}+1}$ is again $M(M^\mathcal{K})$ etc.

PROPOSITION 2.14. i) $\alpha \leq \alpha' \wedge \beta \leq \beta' \Rightarrow C(\alpha,\beta) \subseteq C(\alpha',\beta')$.
ii) $\beta < \pi \Rightarrow |C(\alpha,\beta)| < \pi$.
iii) $\lambda \in Lim \Rightarrow C(\alpha,\lambda) = \bigcup_{\eta<\lambda} C(\alpha,\eta) \wedge C(\lambda,\alpha) = \bigcup_{\eta<\lambda} C(\eta,\alpha)$.
iv) $C(\alpha,\Xi(\alpha)) \cap \mathcal{K} = \Xi(\alpha)$.
v) $C(\alpha,\Psi^\xi_\pi(\alpha)) \cap \pi = \Psi^\xi_\pi(\alpha)$.
vi) $\{\pi : \pi \in M^\alpha \wedge \xi \in C(\alpha,\pi) \cap \alpha\} \subseteq M^\xi$.
vii) M^ξ *stationary in* $\pi \Rightarrow \pi \in M^\xi$.

Let $\mathcal{K}^\Gamma := \min\{\alpha > \mathcal{K} : \forall \xi,\eta < \alpha . \varphi\xi\eta < \alpha\}$.

THEOREM 2.15. M^α *is stationary in* $\mathcal{K}$ *for* $\alpha < \mathcal{K}^\Gamma$.

REMARK. For the proof of this theorem the Π^1_1-indescribability of $\mathcal{K}$ is used. This is the only point where a stronger property than the regularity of $\mathcal{K}$ is needed. All following claims in this section follow from this theorem, the definitions and the propositions above. For a proof see [20].

Corollary. $\alpha \in C(\alpha, \Xi(\alpha))$ and $\Xi(\alpha) < \mathcal{K}$ for $\alpha < \mathcal{K}^\Gamma$.

From now on we only use ordinals $< \mathcal{K}^\Gamma$.

Proposition 2.16. $\Xi(\alpha) < \Xi(\beta)$ *iff* $(\alpha < \beta \wedge \alpha \in C(\beta, \Xi(\beta))) \vee (\beta < \alpha \wedge \beta \notin C(\alpha, \Xi(\alpha)))$

Corollary. $\alpha \neq \beta \Rightarrow \Xi(\alpha) \neq \Xi(\beta)$.

Theorem 2.17. *Let M^ξ stationary in π, $\xi \leq \alpha$ and $\xi, \pi, \alpha \in C(\alpha, \pi)$. Then we have*

$$\Psi^\xi_\pi(\alpha) \in M^\xi \cap \pi.$$

Further: M^ξ is not stationary in $\Psi^\xi_\pi(\alpha)$ for $\xi > 0$.

Theorem 2.18. i) $\Psi^\xi_\pi(\alpha) < \pi \Rightarrow \Psi^\xi_\pi(\alpha) \neq \Xi(\beta)$.
ii) $\Psi^\xi_\pi(\alpha) < \pi \wedge \Psi^\sigma_\kappa(\beta) < \kappa \wedge \Psi^\xi_\pi(\alpha) = \Psi^\sigma_\kappa(\beta) \Rightarrow \alpha = \beta \wedge \pi = \kappa \wedge \xi = \sigma$.

Proposition 2.19.
i) $\gamma \in C(\alpha, \beta)$ *iff* $S(\gamma) \in C(\alpha, \beta)$.
ii) *For $\sigma < \mathcal{K}$ we have*

$$\sigma \in C(\alpha, \beta) \text{ iff } \Omega_\sigma \in C(\alpha, \beta).$$

Proposition 2.20.
i) $0 < \alpha \wedge \pi \in M^\alpha \Rightarrow \Omega_\pi = \pi$.
ii) $\pi \in M^1 \Rightarrow \Omega_{\Psi^0_\pi(\alpha)} = \Psi^0_\pi(\alpha)$.
iii) $\pi = \Omega_{\zeta+1} \wedge \alpha \in C(\alpha, \pi) \Rightarrow \Omega_\zeta < \Psi^0_\pi(\alpha) < \Omega_{\zeta+1}$.
iv) $\Psi^0_\pi(\alpha) < \pi \Rightarrow \Psi^0_\pi(\alpha) \notin Reg$.

Theorem 2.21. *Let $\Psi^\xi_\pi(\alpha) < \pi$ and $\Psi^\sigma_\kappa(\beta) < \pi \cap \kappa$. We have*

$$\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\kappa(\beta)$$

iff
i) $\alpha < \beta \wedge \alpha, \xi, \pi \in C(\beta, \Psi^\sigma_\kappa(\beta)) \wedge \Psi^\xi_\pi(\alpha) < \kappa$,
ii) $\beta \leq \alpha \wedge \{\beta, \sigma, \kappa\} \not\subseteq C(\alpha, \Psi^\xi_\pi(\alpha))$,
iii) $\alpha = \beta \wedge \kappa = \pi \wedge \xi < \sigma \wedge \xi \in C(\beta, \Psi^\sigma_\kappa(\beta))$ *or*
iv) $\sigma < \xi \wedge \sigma \notin C(\xi, \Psi^\xi_\pi(\alpha))$.

Theorem 2.22.

$$\Psi^\xi_\pi(\alpha) < \Xi(\beta) \text{ iff } \pi \leq \Xi(\beta) \vee (\beta < \alpha \wedge \beta \notin C(\alpha, \Psi^\xi_\pi(\alpha)))$$

Definition 2.23. We define inductively a set $\mathcal{T}(K)$ of ordinals and a function

$$m : \mathcal{T}(K) \cap Reg \rightarrow \mathcal{T}(K)$$

by the following rules:
(T1) $0, \mathcal{K} \in \mathcal{T}(K)$
(T2) $S(\alpha) \subseteq \mathcal{T}(K) \Rightarrow \alpha \in \mathcal{T}(K)$
(T3) $\xi \in \mathcal{T}(K) \cap \mathcal{K}$ and $0 < \xi < \Omega_\xi \Rightarrow \Omega_\xi \in \mathcal{T}(K)$ and $m(\Omega_\xi) = 1$ for $\Omega_\xi \in Reg$.
(T4) $\alpha \in \mathcal{T}(K) \cap \mathcal{K}$ and $0 < \alpha \Rightarrow \Xi(\alpha) \in \mathcal{T}(K)$ and $m(\Xi(\alpha)) = \alpha$.
(T5) $\alpha, \xi, \pi \in \mathcal{T}(K) \cap C(\alpha, \pi)$, $\xi \leq \alpha$ and $\xi \in C(m(\pi), \pi) \cap m(\pi) \Rightarrow \Psi^\xi_\pi(\alpha) \in \mathcal{T}(K)$ and for $\xi > 0$ let $m(\Psi^\xi_\pi(\alpha)) = \xi$.

The ordinal $m(\pi)$ is called the Mahlo degree of π.

PROPOSITION 2.24.
i) *For* $\delta \in \mathcal{T}(K)$*:*
  a) $\delta \in C(\mathcal{K}^\Gamma, 0)$.
  b) δ *inaccessible and* $\delta < \mathcal{K} \Rightarrow \delta \in M^{m(\delta)}$ *but* $M^{m(\delta)}$ *not stationary in* δ. *Further:* $m(\delta) = \sup\{\beta : \delta \in M^\beta\}$.
ii) *For* $\pi \in \mathcal{T}(K) \cap Reg$ *and* $\xi \in \mathcal{T}(K)$*:*

$$M^\xi \textit{ stationary in } \pi \textit{ iff } \xi \in C(m(\pi), \pi) \cap m(\pi).$$

iii) *For every ordinal* $\beta \in \mathcal{T}(K)$ *there is a unique representation of* β *with the symbols* 0, $\mathcal{K}$, $+$, φ, Ω, Ξ, Ψ.

DEFINITION 2.25. We define a finite set $K_\delta(\alpha)$ by structural induction over the term $\alpha \in \mathcal{T}(K)$:

(K1) $K_\delta(0) := K_\delta(\mathcal{K}) := \emptyset$
(K2) For $\alpha \notin S$ let $K_\delta(\alpha) := \bigcup_{\beta \in S(\alpha)} K_\delta(\beta)$.
(K3) For $0 < \xi < \Omega_\xi < \mathcal{K}$ let $K_\delta(\Omega_\xi) := K_\delta(\xi)$.
(K4)

$$K_\delta(\Xi(\beta)) := \begin{cases} \emptyset & \text{for } \Xi(\beta) < \delta \\ K_\delta(\beta) \cup \{\beta\} & \text{otherwise} \end{cases}$$

(K5) For $\alpha =_{\mathsf{NF}} \Psi^\sigma_\kappa(\beta)$ let

$$K_\delta(\alpha) := \begin{cases} \emptyset & \text{for } \alpha < \delta \\ K_\delta(\kappa) \cup K_\delta(\sigma) \cup K_\delta(\beta) \cup \{\beta\} & \text{otherwise} \end{cases}$$

PROPOSITION 2.26. *For* $\alpha \in \mathcal{T}(K)$ *we have:*

$$\alpha \in C(\gamma, \delta) \textit{ iff } K_\delta(\alpha) < \gamma.$$

For proofs of the above see [20]. The definitions, propositions and theorems above give us a primitive recursive decision procedure for $\alpha \in \mathcal{T}(K)$ (where α runs over all words over the alphabet $\{0, \mathcal{K}, +, \varphi, \Omega, \Xi, \Psi\}$). By ordinals we mean elements of $\mathcal{T}(K)$ in the following. Note that the propositions and theorems above give us further primitive recursive decision procedures for the following properties: $\alpha < \beta, \alpha \in Lim, \alpha \in H, \alpha \in S, \alpha \in Card, \alpha \in Reg, \alpha$ inaccessible $, \alpha \in M^\xi, \alpha \in C(\gamma, \delta)$ and M^ξ stationary in π.
For the remainder of this paper we understand $\mathcal{T}(K)$ and the corresponding relations as primitive recursive subsets of the natural numbers. We finish this section by giving primitive recursive definitions for some auxiliary operations on $\mathcal{T}(K)$ which we need later. For the operations $+, \varphi, \alpha \mapsto \Omega_\alpha, \Xi$ and Ψ as well as for $\omega^\alpha := \varphi 0 \alpha$ and $\omega_k(\alpha)$ with $\omega_0(\alpha) := \alpha$ and $\omega_{k+1}(\alpha) := \omega^{\omega_k(\alpha)}$ primitive recursive definitions are given by the definitions, propositions and theorems above. The natural sum # is defined with the help of the cantor normal form. To define multiplication $\cdot$ we only need to say what $\alpha \cdot \omega^\beta$ is, since we have $\alpha \cdot 0 = 0$ and $\alpha \cdot (\omega^{\beta_0} + \ldots + \omega^{\beta_n}) = \alpha \cdot \omega^{\beta_0} + \ldots + \alpha \cdot \omega^{\beta_n}$.
Let $\alpha = \omega^{\alpha_0} \cdot a_0 + \ldots + \omega^{\alpha_n} \cdot a_n$ with $\alpha_0 > \ldots > \alpha_n$, $a_0, \ldots, a_n < \omega$. Then we have

$$\alpha \cdot \omega^0 = \alpha$$

and

$$\alpha \cdot \omega^{\beta} = \omega^{\alpha_0} \cdot \omega^{\beta} = \omega^{\alpha_0+\beta} \text{ for } \beta > 0.$$

We define a generalisation of the Veblen function $\hat{\varphi}$ by

$$\hat{\varphi}_0 := id \quad \text{and} \quad \hat{\varphi}_\alpha := \varphi_{\alpha_0} \circ \ldots \circ \varphi_{\alpha_n}$$

for $\alpha =_{\mathsf{NF}} \omega^{\alpha_0} + \ldots + \omega^{\alpha_n}$ and it is easy to see that we have

$$\hat{\varphi}_\alpha(\beta) < \hat{\varphi}(\gamma) \text{ for } \beta < \gamma \text{ and } \hat{\varphi}_{\alpha+\beta} = \hat{\varphi}_\alpha \circ \hat{\varphi}_\beta.$$

The smallest regular cardinal larger than α (the level of α, German: Stufe) can be calculated as follows: For $\alpha \notin S$ let $\mathsf{St}(\alpha) := \mathsf{St}(\max S(\alpha))$. Otherwise let

$$\mathsf{St}(0) := \Omega_1 \qquad \mathsf{St}(\mathcal{K}) := \Omega_{\mathcal{K}+1} \qquad \mathsf{St}(\Omega_\xi) := \Omega_{\xi+1}$$

$$\mathsf{St}(\Xi(0)) := \Omega_1 \quad \mathsf{St}(\Xi(\alpha)) := \Omega_{\Xi(\alpha)+1}$$

$$\mathsf{St}(\Psi^0_\pi(\alpha)) := \Omega_{\Psi^0_\pi(\alpha)+1} \text{ for } \pi \in M^1$$

$$\mathsf{St}(\Psi^0_\pi(\alpha)) := \Omega_{\xi+1} \text{ for } \pi = \Omega_{\xi+1}$$

$$\mathsf{St}(\Psi^\xi_\pi(\alpha)) := \Omega_{\Psi^\xi_\pi(\alpha)+1} \text{ for } \xi > 0.$$

Let $\pi^- := \Omega_\xi$ for $\pi = \Omega_{\xi+1}$. Furthermore we abbreviate $\alpha_n \leq \beta_0$ for $\alpha =_{\mathsf{NF}} \omega^{\alpha_0} + \ldots + \omega^{\alpha_n}$ and $\beta =_{\mathsf{NF}} \omega^{\beta_0} + \ldots + \omega^{\beta_m}$ by $\mathsf{NF}(\alpha, \beta)$. This relation between α and β is used later to conclude $\alpha \in C(\gamma, \delta)$ from $\alpha + \beta \in C(\gamma, \delta)$. For $\mu \in Card$ let

$$\bar{\mu} := \begin{cases} \mu + 1 & \text{if } \mu \in Reg \cup \{\mathcal{K}\} \\ \mu & \text{otherwise} \end{cases}.$$

§3. The language $\mathcal{L}_{RS}$ of ramified set theory. In this section we recall the definition of the language of ramified set theory $\mathcal{L}_{RS}$. Let $\mathcal{L}_{\mathrm{Ad}}$ be the first order language of set theory without negation built up from the 2-ary predicate symbols $\in, \notin$ and predicate symbols $\mathrm{Ad}^\xi, \neg\mathrm{Ad}^\xi$ of arity one for $\xi \in \mathcal{T}(K)$. The negation $\neg\phi$ of a formula ϕ is defined by the de Morgan laws. The language $\mathcal{L}_{RS}$ of ramified set theory is gained from $\mathcal{L}_{\mathrm{Ad}}$ by adding elements of the constructible hierarchy as terms. We write $\phi(x_0, \ldots, x_n)$ for $\mathsf{FV}(\phi) \subseteq \{x_0, \ldots, x_n\}$ and $\phi(t)$ to emphasise the substitution of t in ϕ for a variable x. Given a term t we write ϕ^t for the formula obtained from ϕ by replacing in ϕ every unbounded quantifier $\forall x, \exists x$ by $\forall x \in t, \exists x \in t$. We start by defining the RS-terms $\mathcal{T}$:

Definition 3.1 (RS-term).
1. For $\alpha \in \mathcal{T}(K)$ let $L_\alpha \in \mathcal{T}$ of stage α.
2. $\phi(x_0, \ldots, x_n) \in \mathcal{L}_{\mathrm{Ad}}$ and $a_1, \ldots, a_n \in \mathcal{T}$ of stages $< \alpha$ then

$$[x \in L_\alpha : \phi(x, a_1, \ldots, a_n)^{L_\alpha}] \in \mathcal{T}$$

of stage α.

We write $\mathsf{stg}(t)$ for the stage of a term t. Let $\mathcal{T}_\alpha := \{t \in \mathcal{T} | \mathsf{stg}(t) < \alpha\}$. Note that there are no free variables in RS-terms. For $t \equiv [x \in L_\alpha : \phi(x, a_1, \ldots, a_n)^{L_\alpha}]$ we call ϕ the skeleton of t and the number of logical symbols in ϕ is called the outer rank t. We obtain RS-formulas from Δ_0-formulas of the language $\mathcal{L}_{\mathrm{Ad}}$ by substitution of RS-terms for free variables:

DEFINITION 3.2 (RS-formula).

1. $u \in v, u \notin v, \mathrm{Ad}^\xi(v), \neg\mathrm{Ad}^\xi(v)$ are RS-formulas for $\xi \in \mathcal{T}(K)$ and $u, v \in \mathcal{T} \cup Var$.
2. A, B RS-formulas $\Rightarrow A \wedge B$ and $A \vee B$ RS-formulas.
3. If A is an RS-formula and $x \in Var \setminus \{u\} \Rightarrow \forall x(x \notin u \vee A)$ and $\exists x(x \in u \wedge A)$ RS-formulas.

Skeleton and outer rank are defined analogously as by terms. The negation of a formula is again defined by the de Morgan laws. We use the standard notations $A \to B$ for $\neg A \vee B$, $\forall x \in vB$ for $\forall x(x \in v \to B)$ etc. In particular we write $u \subseteq v$ for $\forall x \in u.x \in v$, $u = v$ for $u \subseteq v \wedge v \subseteq u$, $\mathrm{tran}(u)$ for $\forall x \in u.x \subseteq u$ and $\mathrm{infin}(u)$ for $\exists x \in u(x \subseteq x) \wedge \forall x \in u \forall y \in u \exists z \in u(x \in z \wedge y \in z)$. Uppercase Latin letters are mainly used for RS-formulas whereas the Greek letters ϕ, ψ, χ are used for $\mathcal{L}_{\mathrm{Ad}}$-formulas. For RS-terms s, t let $A^{(s,t)}$ denote the RS-formula gained from A by replacing in A every t bounded quantifier $Qx \in t$ by $Qx \in s$. We write $A^{(s,\alpha)}$ instead of $A^{(s,L_\alpha)}$ and $A^{(\beta,\alpha)}$ instead of $A^{(L_\beta,L_\alpha)}$. We use the abbreviation Qx^α for bounded quantifiers $Qx \in L_\alpha$. The bracketed inequality $[u \neq v]$ stands for $u \neq v$ or $v \neq u$.

DEFINITION 3.3. For θ RS-term or RS-formula let

$$\mathsf{k}(\theta) := \{\alpha \in \mathcal{T}(K) : L_\alpha \text{ occurs in } \theta\} \qquad \mathsf{lev}(\theta) := \max(\mathsf{k}(\theta) \cup \{0\})$$

For technical reasons we set $\mathsf{k}(0) := \mathsf{k}(1) := \emptyset$ and $\mathsf{lev}(0) := \mathsf{lev}(1) := 0$, where $0, 1$ are not viewed as ordinals and $\mathsf{k}(\alpha) := \{\alpha\}$ for $\alpha \in \mathcal{T}(K)$. We have $\mathsf{stg}(t) = \mathsf{lev}(t)$ for RS-terms t and therefore

$$\mathcal{T}_\alpha = \{t \in \mathcal{T} : \mathsf{lev}(t) < \alpha\}.$$

We write $\mathcal{T}_t$ for $\mathcal{T}_{\mathsf{lev}(t)}$.

DEFINITION 3.4. For RS-terms a, b where $\mathsf{lev}(a) < \mathsf{lev}(b)$ let

$$a \overset{o}{\in} b :\equiv \begin{cases} B(a) & \text{for } b \equiv [x \in L_\beta : B(x)] \\ \top & \text{for } b \equiv L_\beta \end{cases}$$

$$a \overset{o}{\notin} b :\equiv \neg(a \overset{o}{\in} b), \text{ where } \neg\top :\equiv \bot$$

and $\top, \bot$ are not RS-formulas. We define $\top \to A :\equiv \top \wedge A :\equiv \bot \vee A :\equiv A$.

DEFINITION 3.5. We can view every RS-sentence A as a (possibly infinite) conjunction $\bigwedge(A_i)_{i \in J}$ or disjunction $\bigvee(A_i)_{i \in J}$ of RS-sentences. We write $A \simeq \bigwedge(A_i)_{i \in J}$, $A \simeq \bigvee(A_i)_{i \in J}$ respectively for this relationship which is defined by:

1. $\mathrm{Ad}^\alpha(t) :\simeq \bigvee(L_\rho = t)_{\rho \in M^\alpha \cap (\mathsf{lev}(t)+1)}$
2. $a \in b :\simeq \bigvee(t \overset{o}{\in} b \wedge t = a)_{t \in \mathcal{T}_b}$
3. $\exists x \in bA(x) :\simeq \bigvee(t \overset{o}{\in} b \wedge A(t))_{t \in \mathcal{T}_b}$
4. $(A_0 \vee A_1) :\simeq \bigvee(A_i)_{i \in \{0,1\}}$
5. $\neg A :\simeq \bigwedge(\neg A_i)_{i \in J}$ for $A \simeq \bigvee(A_i)_{i \in J}$.

Remark. In the following we understand every RS-sentence A as such a conjunction or disjunction.

We map an ordinal $\mathcal{T}(K)$ to every RS-term and every RS-sentence:

Definition 3.6. We define $\mathsf{rk}(\theta)$ for RS-terms and RS-formulas by primitive recursion over the number of logical symbols occurring in θ:
1. $\mathsf{rk}(L_\alpha) := \omega \cdot \alpha$
2. $\mathsf{rk}([x \in L_\alpha : A(x)]) := \max\{\omega \cdot \alpha + 1, \mathsf{rk}(A(L_0)) + 2\}$
3. $\mathsf{rk}(\mathrm{Ad}^\xi(a)) := \mathsf{rk}(\neg\mathrm{Ad}^\xi(a)) := \mathsf{rk}(a) + 5$
4. $\mathsf{rk}(a \in b) := \mathsf{rk}(a \notin b) := \max\{\mathsf{rk}(a) + 6, \mathsf{rk}(b) + 1\}$
5. $\mathsf{rk}(\exists x \in b A(x)) := \mathsf{rk}(\forall x \in b A(x)) := \max\{\mathsf{rk}(b), \mathsf{rk}(A(L_0)) + 2\}$
6. $\mathsf{rk}(A \wedge B) := \mathsf{rk}(A \vee B) := \max\{\mathsf{rk}(A), \mathsf{rk}(B)\} + 1$

We use the following properties of the rank $\mathsf{rk}(A)$ throughout this paper:

Proposition 3.7. *Let* $A \simeq \bigvee(A_i)_{i\in J}$ *or* $A \simeq \bigwedge(A_i)_{i\in J}$. *Then*
a) *there is an* $n \in \omega$ *with* $\mathsf{rk}(A) = \omega \cdot \mathsf{lev}(A) + n$,
b) $\mathsf{rk}(A_i) < \mathsf{rk}(A)$ *for* $i \in J$,
c) $\mathsf{k}(A_i) \subseteq \mathsf{k}(A) \cup \mathsf{k}(i)$ *for* $i \in J$,
d) $\mathsf{rk}(A) = \omega \cdot \alpha \Rightarrow A \equiv \exists x \in L_\alpha B(x)$ *or* $A \equiv \forall x \in L_\alpha B(x)$.
e) $\mathsf{rk}(A) = \mathsf{rk}(\neg A)$.

Proof. See [8]. ⊣

An RS-formula A is an element of $\Delta_0(\alpha)$ if $\mathsf{k}(A) \subseteq \alpha$. An RS-formula is an element of $\Pi_k(\alpha)$ of the form

$$\forall x_1 \in L_\alpha \dots Qx_k \in L_\alpha F(x_1, \dots, x_k)$$

where $\forall x_1, \dots, Qx_k$ are alternating quantifiers and $F(L_0, \dots, L_0)$ is a $\Delta_0(\alpha)$-formula. $\Sigma_k(\alpha)$ is defined analogously.

§4. Inferences, derivations, proof systems. We call finite sets of RS-formulas sequents and use the uppercase Greek letters Γ, Γ', Δ for sequents. We use as well the following notations for sequents: $A_1, \dots, A_n$ for $\{A_1, \dots, A_n\}$, A, Γ, Γ' for $\{A\} \cup \Gamma \cup \Gamma'$ etc. We define

$$\mathsf{k}(\Gamma) := \bigcup_{A\in\Gamma} \mathsf{k}(A).$$

Definition 4.1 (Inference). An inference $\mathcal{I}$ consists of
1. an index set $|\mathcal{I}|$ (for the premises of $\mathcal{I}$),
2. a sequent $\Delta(\mathcal{I})$ (the principal formulas of $\mathcal{I}$),
3. a family of sequents $(\Delta_i(\mathcal{I}))_{i\in|\mathcal{I}|}$ (the minor formulas of $\mathcal{I}$),
4. a set $\mathsf{Eig}(\mathcal{I})$ which is either empty or a singleton $\{y\}$ where $y \notin \mathsf{FV}(\Delta(\mathcal{I}))$ (y is called the eigenvariable of $\mathcal{I}$),
5. a finite set $\mathsf{k}(\mathcal{I}) \subseteq \mathcal{T}(K)$.

We define derivations by induction:

Definition 4.2 (Derivation).
If $\mathcal{I}$ is an inference and $(d_i)_{i\in|\mathcal{I}|}$ a family of derivations with

1.
$$\mathsf{End}(d) := \Delta(\mathcal{I}) \cup \bigcup_{i \in |\mathcal{I}|} (\mathsf{End}(d_i) \setminus \Delta_i(\mathcal{I})) \text{ finite}$$

2. $\mathsf{Eig}(\mathcal{I}) \cap \mathsf{FV}(\mathsf{End}(d)) = \emptyset$

then $d := \mathcal{I}(d_i)_{i \in |\mathcal{I}|}$ is a derivation with $\mathsf{depth}(d) := \sup\{\mathsf{depth}(d_i) + 1 : i \in |\mathcal{I}|\}$. $\mathsf{End}(d)$ is called the endsequent of d.

An inference $\mathcal{I}$ is called finitary, if $|\mathcal{I}| = \{0, \dots, n-1\} \in \omega$. A derivation d is called finitary if all inferences in d are finitary.

Notations. 1.
$$(\mathcal{I})\frac{\dots \Delta_i \dots (i \in J)}{\Delta}[!y!]$$
for $\mathcal{I}$ is an inference where $|\mathcal{I}| = J, \Delta(\mathcal{I}) = \Delta, \Delta_i(\mathcal{I}) = \Delta_i$ and $\mathsf{Eig}(\mathcal{I}) = \emptyset$, $\mathsf{Eig}(\mathcal{I}) = \{y\}$ respectively.

2. If $|\mathcal{I}| = \{0, \dots, n\}$ we write
$$(\mathcal{I})\frac{\Delta_0 \dots \Delta_n}{\Delta}$$
and if $|\mathcal{I}| = \emptyset$ we write
$$(\mathcal{I})\Delta.$$

3. Inferences with $|\mathcal{I}| = \emptyset$ are called axioms or atomic derivations.
4. If $|\mathcal{I}| = \{0, \dots, n\}$ we write $d = \mathcal{I} d_0 \dots d_n$ instead of $d = \mathcal{I}(d_i)_{i \in |\mathcal{I}|}$.

Remark. If $d = \mathcal{I}(d_i)_{i \in |\mathcal{I}|}$ is a derivation where $\Delta(\mathcal{I}) \subseteq \Gamma$ and for every $i \in |\mathcal{I}|$ we have $\mathsf{End}(d_i) \subseteq \Gamma \cup \Delta_i(\mathcal{I})$, then $\mathsf{End}(d) \subseteq \Gamma$.

Definition 4.3. Let $d = \mathcal{I} d_0 \dots d_{n-1}$ be a finitary derivation. Then let
$$\mathsf{k}(d) := \mathsf{k}(\mathcal{I}) \cup \bigcup_{i<n} \mathsf{k}(d_i), \quad \mathsf{k}_c(d) := (\mathsf{k}(\mathcal{I}) \setminus \mathsf{k}(\Delta(\mathcal{I}))) \cup \bigcup_{i<n} \mathsf{k}_c(d_i).$$

Proposition 4.4. *If d is a finitary derivation such that $\mathsf{k}(\Delta_i(\mathcal{I})) \subseteq \mathsf{k}(\mathcal{I}) \cup \mathsf{k}(\Delta(\mathcal{I}))$ for every inference $\mathcal{I}$ in d and every $i \in |\mathcal{I}|$, then $\mathsf{k}(d) \subseteq \mathsf{k}(\mathsf{End}(d)) \cup \mathsf{k}_c(d)$.*

Proof. See [8]. ⊣

Definition 4.5 (Proof system).
A proof system is a class of inferences. A proof system $\mathcal{S}$ is called finitary if all inferences of $\mathcal{S}$ are finitary. A derivation d is called a $\mathcal{S}$-derivation if all inferences of d are inferences of $\mathcal{S}$.

§5. The proof system $RS(\mathcal{K})$. The validity relation in a structure $\langle M, \dots \rangle$ can be seen as an (possibly infinitary) proof system: A formula $\forall x \phi(x)$ is valid in the structure if all premises $\phi(a)$ for $a \in M$ are valid in the structure. This calculus is sound in the sense that we infer valid formulas from valid premises. It remains sound if we add further inferences which are valid in the structure. The structure in question here is an initial segment of the constructible hierarchy. The inferences Cut_C and $\mathrm{Ref}_{\mathcal{K}} A$ are used to embed the finitary into the infinitary calculus. A (infinitary) derivation is called cut free if it contains no inference Cut_C. Cut free derivations have the property that all formulas occurring in the derivation are Gentzen sub formulas of formulas in the endsequent. This important property which is called the sub formula property will be used later. Take a look at the infinitary proof system consisting of the inferences $\bigwedge_A, \bigvee_A^{i_0}$ and Cut_C. Thanks to the symmetry of the other inferences cuts are easy to eliminate. However to embed Π_3-Reflection we add the inference $\mathrm{Ref}_{\mathcal{K}}A$. To eliminate cuts of rank $\mathcal{K}$ we use the inference $\mathrm{Ref}^{\xi}_{\pi}A(s)$. This rule allows to eliminate $\mathrm{Ref}_{\mathcal{K}}A$ inferences from certain derivations by replacing them by $\mathrm{Ref}^{\xi}_{\pi}A(s)$-inferences which can be treated in an easier way (see [21]).

Inferences of $RS(\mathcal{K})$:

$$(\textstyle\bigwedge_A) \qquad \frac{\cdots A_i \cdots (i \in J)}{A} \qquad \text{if } A \simeq \textstyle\bigwedge(A_i)_{i\in J}$$

$$(\textstyle\bigvee_A^{i_0}) \qquad \frac{A_{i_0}}{A} \qquad \text{if } A \simeq \textstyle\bigvee(A_i)_{i\in J} \text{ and } i_0 \in J$$

$$(\mathrm{Cut}_C) \qquad \frac{C \quad \neg C}{\emptyset}$$

$$(\mathrm{Ref}_{\mathcal{K}}A) \qquad \frac{A}{\exists z \in L_{\mathcal{K}}(\mathrm{tran}(z) \wedge z \neq \emptyset \wedge A^{(z,\mathcal{K})})} \qquad \text{if } A \in \Pi_3(\mathcal{K})$$

$$(\mathrm{Ref}^{\xi}_{\pi}A(s)) \qquad \frac{A(s)}{\exists z \in L_{\pi}(\mathrm{Ad}^{\xi}(z) \wedge \exists u \in zA(u)^{(z,\pi)})}$$

if $A(s) \in \Pi_2(\pi)$ and $\xi \in C(m(\pi),\pi) \cap m(\pi)$

$$(\mathrm{Rep}_{i_0}) \qquad \frac{\emptyset}{\emptyset} \qquad \text{with } |\mathrm{Rep}_{i_0}| := \{i_0\}$$

$$\mathsf{k}(\mathcal{I}) := \begin{cases} \mathsf{k}(\Delta(\mathcal{I})) \cup \mathsf{k}(i_0) & \text{if } \mathcal{I} = \bigvee_A^{i_0} \\ \mathsf{k}(C) & \text{if } \mathcal{I} = \mathrm{Cut}_C \\ \mathsf{k}(\Delta(\mathcal{I})) & \text{otherwise} \end{cases}$$

Remark. Note that we have $\mathsf{k}(\Delta(\mathcal{I})) \subseteq \mathsf{k}(\mathcal{I})$.

§6. Notation systems. An $RS(\mathcal{K})$-derivation is generally not a finite object. To argue from a finitary point of view we introduce notations for some $RS(\mathcal{K})$-derivations. The notations are finite objects unlike the infinitary derivations they denote. We work then with these notations only. We slightly modify the notion in [8] to define what it means to be a notation system for $RS(\mathcal{K})$-derivations:

Definition 6.1 (Notation system).
A notation system for $RS(\mathcal{K})$-derivations consists of a non empty set $\mathcal{D}$ of finitary derivations d with $\mathsf{FV}(\mathsf{End}(d)) = \emptyset$ and maps

$\mathsf{o}, \mathsf{deg}, \mathsf{ref} : \mathcal{D} \to On$,
$\mathsf{tp} : \mathcal{D} \to RS(\mathcal{K})$,
$[] : \{(d, i) : d \in \mathcal{D}$ and $i \in |\mathsf{tp}(d)|\} \to \mathcal{D}$ (we write $d[i]$ for $[](d, i)$),
such that for every $d \in \mathcal{D}$:

a) $\Delta(\mathsf{tp}(d)) \subseteq \mathsf{End}(d)$
b) $\mathsf{End}(d[i]) \subseteq \mathsf{End}(d), \Delta_i(\mathsf{tp}(d))$ for every $i \in |\mathsf{tp}(d)|$
c) $\mathsf{o}(d[i]) < \mathsf{o}(d)$ for every $i \in |\mathsf{tp}(d)|$
d) $\mathsf{deg}(d[i]) \le \mathsf{deg}(d)$ for every $i \in |\mathsf{tp}(d)|$
e) $\mathsf{ref}(d[i]) \le \mathsf{ref}(d)$ for every $i \in |\mathsf{tp}(d)|$
f) $\mathsf{tp}(d) = \mathrm{Cut}_C \Rightarrow \mathsf{rk}(C) < \mathsf{deg}(d)$
g) $\mathsf{tp}(d) = \bigvee_A^{i_0} \Rightarrow \mathsf{k}(i_0) < \mathsf{o}(d)$
h) $\mathsf{tp}(d) = \bigwedge_A \Rightarrow \mathsf{k}(i) < \mathsf{o}(d)$ for every $i \in |\mathsf{tp}(d)|$
i) $\mathsf{tp}(d) = \mathrm{Ref}_{\mathcal{K}} A \Rightarrow \mathcal{K} < \mathsf{o}(d)$.
j) $\mathsf{tp}(d) = \mathrm{Ref}_\tau^\sigma A(s) \Rightarrow \sigma < \mathsf{ref}(d)$ and $\tau, \mathsf{o}(d[0]){+}1 < \mathsf{o}(d)$ and $\sigma \in C(m(\tau), \tau) \cap m(\tau)$.

A notation system for $RS(\mathcal{K})$-derivations is called normal if

k) $\mathsf{k}(\mathsf{End}(d)) \subseteq \mathsf{k}(d)$ for every $d \in \mathcal{D}$.

A notation system for $RS(\mathcal{K})$-derivation is controlled by the operator $\mathcal{H} : \mathcal{P}(On) \to \mathcal{P}(On)$, if for every $d \in \mathcal{D}$:

l) $\mathsf{k}(\mathsf{tp}(d)) \cup \{\mathsf{o}(d)\} \subseteq \mathcal{H}(\mathsf{k}(d))$
m) $\mathsf{k}(d[i]) \cup \{\mathsf{o}(d[i])\} \subseteq \mathcal{H}(\mathsf{k}(d) \cup \mathsf{k}(i))$ for every $i \in |\mathsf{tp}(d)|$.

The inference $\mathsf{tp}(d)$ is the last inference of the derivation denoted by d, $d[i]$ is a notation of the sub derivation at place i, o is the ordinal assignment and $\mathsf{deg}(d)$, $\mathsf{ref}(d)$ are bounds for the Cut, Π_2-Reflection inferences occurring in d respectively. The definition is almost the same as in [8]. However a few things are added: The function $\mathsf{ref} : \mathcal{D} \to On$ together with the conditions e),j) which say: For all Ref_τ^σ-inferences occurring in the derivation denoted by d is $\mathsf{ref}(d)$ an upper bound for σ and the σ-Mahlo numbers are stationary in τ. Further we introduced the conditions h),i),k) and the requirement

$$\mathsf{o}(d[i]) \in \mathcal{H}(\mathsf{k}(d) \cup \mathsf{k}(i)).$$

The concept of operator controlled derivations is as well due to Buchholz [4].
If $(\mathcal{D}, \mathsf{o}, \mathsf{deg}, \mathsf{ref}, \mathsf{tp}, [])$ is a notation system for $RS(\mathcal{K})$-derivations then we can unfold $d \in \mathcal{D}$ by transfinite recursion over $\mathsf{o}(d)$ to an infinitary $RS(\mathcal{K})$-derivation

$$d^\infty := \mathsf{tp}(d)(d[i]^\infty)_{i \in |\mathsf{tp}(d)|}.$$

It is easy to see that d^∞ is a $RS(\mathcal{K})$-derivation with

$$\mathsf{End}(d^\infty) \subseteq \mathsf{End}(d), \mathsf{depth}(d^\infty) \le \mathsf{o}(d), \mathsf{rk}(C) < \mathsf{deg}(d) \text{ for every } \mathrm{Cut}_C \text{ in } d^\infty$$

and g)-j) accordingly. In the following we use finitary notations of infinitary derivations only and make use of primitive recursion instead of transfinite recursion.

§7. The notation system $\mathbf{RS^0}$. We are going to define a notation system to embed all axioms of Π_3-Reflection. In other words our goal is to have for each axiom a notation for a $RS(\mathcal{K})$-derivation which endsequent is the axiom in question. We proceed again similiar to Buchholz [8].

Definition 7.1 (Axiom system for Π_3-Reflection).

(Ext)	$\forall x \forall y \forall z (x = y \to (x \in z \to y \in z))$
(Found)	$\forall \vec{z} \forall x_0 (\forall x (\forall y \in x. \varphi(y, \vec{z}) \to \varphi(x, \vec{z})) \to \varphi(x_0, \vec{z}))$
(Pair)	$\forall x \forall y \exists z (x \in z \wedge y \in z)$
(Union)	$\forall x \exists y \forall z \in x (z \subseteq y)$
(Infin)	$\forall x \exists y \text{infin}(y)$
(Δ_0-Sep)	$\forall \vec{z} \forall w \exists y (\forall x \in y (x \in w \wedge \varphi(x, \vec{z})) \wedge \forall x \in w (\varphi(x, \vec{z}) \to x \in y))$ for $\varphi \in \Delta_0$
(Π_3-Ref)	$\forall \vec{z} (\varphi(\vec{z}) \to \exists x (\text{tran}(x) \wedge x \neq \emptyset \wedge \varphi(\vec{z})^x))$ for $\varphi \in \Pi_3$

Definition 7.2 (The set of sequents $\mathcal{A}X^0$). Let $\mathcal{A}X^0$ the set of finite sequents of RS-sentences which are given by the schemes 1-17:

(1) $(\forall x_k \varphi(a_0, \dots, a_{k-1}, x_k))^\lambda$
if $a_0, \dots, a_{k-1} \in \mathcal{T}_\lambda$ and $\forall x_0 \dots x_k \varphi(x_0, \dots, x_k)$ an Axiom (Ext), (Found), (Pair), (Union), (Δ_0-Sep) with $\lambda \in Lim$ or (Infin) with $\lambda \in Reg$.
(2) $\forall z_k \in L_\mathcal{K} (\varphi(a_0, \dots, a_{k-1}, z_k)^\mathcal{K} \to \exists x \in L_\mathcal{K} (\text{tran}(x) \wedge x \neq \emptyset \wedge \varphi(a_0, \dots, a_{k-1}, z_k)^x)$
if $a_0, \dots, a_{k-1} \in \mathcal{T}_\mathcal{K}$ and $\forall \vec{z} (\varphi(\vec{z}) \to \exists x (\text{tran}(x) \wedge x \neq \emptyset \wedge \varphi(\vec{z})^x))$ an axiom (Π_3-Ref).
(3) $a = a$
(4) $a \subseteq a$
(5) $b \subseteq L_\alpha$ if $\mathsf{lev}(b) \leq \alpha$
(6) $\forall x \in a (x \subseteq L_\alpha)$ if $\mathsf{lev}(a) \leq \alpha + 1$
(7) $\forall x \in b F(x)$ if $b \equiv [x \in L_\beta : F(x)]$
(8) $\forall x \in a (F(x) \to x \in b)$ if $b \equiv [x \in L_\beta : x \in a \wedge F(x)]$
(9) $\exists x \in L_\alpha (\forall y \in x A(y) \wedge \neg A(x)), \forall x \in a A(x)$ if $\mathsf{lev}(a) \leq \alpha$
(10) $[s_1 \neq t_1], \dots, [s_n \neq t_n], \neg A(\vec{s}), A(\vec{t})$ if every variable from $\vec{x}$ occurs at most once in $A(\vec{x})$
(11) $[s_1 \neq t_1], \dots, [s_n \neq t_n], a \notin t_n, \neg B(s_1, \dots, s_{n-1}, a), A(\vec{t})$ if $A(\vec{x}) \equiv \exists y \in x_n B(x_1, \dots, x_{n-1}, y)$ and every variable from $\vec{x}$ occurs at most once in $A(\vec{x})$
(12) $[s_1 \neq t_1], a \notin t_2, a \neq s_1, t_1 \in t_2$
(13) $a \not\subseteq b, L_\beta \notin a$ if $\mathsf{lev}(b) \leq \beta$
(14) $L_\beta \not\subseteq s, s \notin b$, if $\mathsf{lev}(b) \leq \beta$
(15) $\forall x \in L_\omega \exists u \in L_\omega (\exists y \in u (x \in y) \wedge \mathcal{A}(u))$ with $\mathcal{A}(u) :\equiv \forall x \in u (\forall y \in x (y \neq y) \vee \mathcal{B}(u, x))$ and $\mathcal{B}(u, x) :\equiv \exists x_0 \in u (x_0 \in x \wedge \forall y \in x (y \subseteq x_0))$
(16) $\mathcal{A}(a_n)$ with $a_n := [x \in L_{n+1} : x = L_0 \vee \dots \vee x = L_n]$ where $\mathcal{A}$ as in (16)
(17) $\forall y \in L_0 (y \neq y)$

Let $\Pi = (A_1, \dots, A_n)$.

$$\mathsf{o}(\Pi) := \begin{cases} \omega^{\mathsf{rk}(\varphi(\vec{a})^\lambda)} \# \omega \cdot \lambda & \text{if } \Pi = (\varphi(\vec{a})^\lambda) \text{ of the kind (1)}(Found) \\ \omega^{\mathsf{rk}(\forall x \in a A(x))} \# \omega \cdot \mathsf{lev}(a) & \text{if } \Pi = (F, \forall x \in aA) \text{ of the kind (9)} \\ \omega^{\mathsf{rk}(A_1)} \# \ldots \# \omega^{\mathsf{rk}(A_n)} & \text{otherwise} \end{cases}$$

$$\mathsf{deg}(\Pi) := \begin{cases} \omega \cdot 2 & \text{if } \Pi \text{ of the kind (15) or (16)} \\ 0 & \text{otherwise.} \end{cases}$$

For $\Pi = (A_1, \ldots, A_n) \in \mathcal{A}X^0$ of kind (j) (j=1-17) we define an inference $\mathrm{Ax}^*_j\Pi$ by $|\mathrm{Ax}^*_j\Pi| := \emptyset$, $\Delta(\mathrm{Ax}^*_j\Pi) := \{A_1, \ldots, A_n\}$, $\mathsf{k}(\mathrm{Ax}^*_j\Pi) := \mathsf{k}(\Delta(\mathrm{Ax}^*_j\Pi))$.

Definition 7.3 (The finitary proof system RS^0).
Formulas: RS-sentences
Inferences: $\mathrm{Ax}^*_j\Pi$ and $\bigwedge_{A_0 \wedge A_1}$, $\bigvee^{i_0}_A$, Cut_C, $\mathrm{Ref}_{\mathcal{K}}A$

Definition 7.4 (The notation system $\mathbf{RS^0}$).
$\mathbf{RS^0} := (\mathcal{D}_0, \mathsf{o}, \mathsf{deg}, \mathsf{ref}, \mathsf{tp}, [])$ with $\mathcal{D}_0$:= set of all RS^0-derivations and

$$\mathsf{o}(\mathcal{I}d_0 \ldots d_n) := \begin{cases} \mathsf{o}(\Pi) & \text{if } \mathcal{I} = \mathrm{Ax}^*_j\Pi \\ \max\{\mathsf{o}(d_0), \mathsf{lev}(i_0)\} + 1 & \text{if } \mathcal{I} = \bigvee^{i_0}_A \\ \max\{\mathsf{o}(d_0), \mathcal{K}\} + 1 & \text{if } \mathcal{I} = \mathrm{Ref}_{\mathcal{K}}A \\ \max\{\mathsf{o}(d_0), \ldots, \mathsf{o}(d_n)\} + 1 & \text{otherwise} \end{cases}$$

$$\mathsf{deg}(\mathcal{I}d_0 \ldots d_n) := \begin{cases} \mathsf{deg}(\Pi) & \text{if } \mathcal{I} = \mathrm{Ax}^*_j\Pi \\ \max\{\mathsf{deg}(d_0), \mathsf{deg}(d_1), \mathsf{rk}(C) + 1\} & \text{if } \mathcal{I} = \mathrm{Cut}_C \\ \max\{\mathsf{deg}(d_0), \ldots, \mathsf{deg}(d_n)\} & \text{otherwise} \end{cases}$$

$\mathsf{ref}(\mathcal{I}d_0 \ldots d_n) = 0$.

For $d = \mathcal{I}d_0 \ldots d_n$ with $\mathcal{I} \neq \mathrm{Ax}^*_j\Pi$ let $\mathsf{tp}(d) := \mathcal{I}$ and $d[i] := d_i$.

For $d = \mathrm{Ax}^*_j\Pi$ let $\mathsf{tp}(d) := \bigwedge_A$ where A is the $\bigwedge$-formula A_i with the smallest index i in $\Pi = (A_1, \ldots, A_n)$. Let $d[i]$ be defined similar to [8].
Compared to [8] we have only introduced $d = \mathrm{Ax}^*_2\Pi$ where Π is the sequent

$$(\forall z \in L_{\mathcal{K}}(\varphi(a_0, \ldots, a_{n-1}, z) \rightarrow \exists x \in L_{\mathcal{K}}(\mathrm{tran}(x) \wedge x \neq \emptyset \wedge \varphi(a_0, \ldots, a_{n-1}, z)^x))$$

and $a_0, \ldots, a_{n-1} \in \mathcal{T}_K$.
For this case let

$$d[a_n] := \bigvee_{A_{a_n}} \mathrm{Ref}_{\mathcal{K}}\varphi(a_0, \ldots, a_n)^{\mathcal{K}} \mathrm{Ax}^*_{10}(\neg\varphi(a_0, \ldots, a_n)^{\mathcal{K}}, \varphi(a_0, \ldots, a_n)^{\mathcal{K}})$$

for $a_n \in \mathcal{T}_K$ and $A_{a_n} :\equiv \varphi(a_0, \ldots, a_n)^{\mathcal{K}} \rightarrow \exists x \in L_{\mathcal{K}}(\mathrm{tran}(x) \wedge x \neq \emptyset \wedge \varphi(a_0, \ldots, a_n)^x)$

Theorem 7.5. $\mathbf{RS^0}$ *is a normal notation system for* RS($\mathcal{K}$)- *derivations and is controlled by operators which are closed under the functions* $\lambda x, y.x\#y, \lambda x.\omega \cdot x, \lambda x.\omega^x$ *and* $\lambda x.\mathsf{St}(x)$.

PROOF. Most of the work is already done in [8]. We just have to verify the requirements for notation systems for $d = (\mathrm{Ref}_{\mathcal{K}})Ad_0$ and $d = \mathrm{Ax}_1^*\Pi$ where Π comes from a Π_3-Ref-axiom and to verify for all derivations the condition $\mathsf{o}(d[i]) \in \mathcal{H}(\mathsf{k}(d) \cup \mathsf{k}(i))$. The conditions e) and j) are trivial and the conditions h) and i) are easy to verify. It is easy to see that $\mathbf{RS^0}$ is normal. The proofs are left to the reader. ⊣

§8. The notation system $\mathbf{RS}^+$. In this section we define finitary proof systems RS^λ to get notations for $RS(\mathcal{K})$-derivations which proof the logically valid formulas of first order logic. This systems together with the notation system $\mathbf{RS^0}$ and the cut inference give an embedding of $\mathbf{KP} + \Pi_3$-Reflection.

DEFINITION 8.1 (RS_λ-formula).

1. If $u, v \in \mathcal{T}_\lambda \cup Var, \xi \in On$ then $u \in v, u \notin v, \mathrm{Ad}^\xi(u), \neg\mathrm{Ad}^\xi(u)$ are RS_λ-formulas.
2. If A, B are RS_λ-formulas then $A \wedge B, A \vee B, \forall x \in L_\lambda A, \exists x \in L_\lambda A$ are RS_λ-formulas.
3. If A, B are RS_λ-formulas and $x \neq u, u \in \mathcal{T}_\lambda \cup Var$ then $\forall x \in uA, \exists x \in uA$ are RS_λ-formulas.

DEFINITION 8.2. $\mathsf{rk}_0(A)$ for A RS_λ-formula

1. $\mathsf{rk}_0(A) = \mathsf{rk}_0(\neg A) = 0$ for A atomic
2. $\mathsf{rk}_0(A \wedge B) = \mathsf{rk}_0(A \vee B) = \max\{\mathsf{rk}_0(A), \mathsf{rk}_0(B)\} + 1$
3. $\mathsf{rk}_0(\forall x \in aA) = \mathsf{rk}_0(\exists x \in aA) = \mathsf{rk}_0(A) + 2$

PROPOSITION 8.3. $\mathsf{rk}(A) < \lambda + \mathsf{rk}_0(A)$ *for* $\lambda = \omega^\lambda$ *and* A RS_λ*-sentence.*

PROOF. See [8]. ⊣

DEFINITION 8.4. The finitary proof system RS^λ consists of the inferences:

$(\mathrm{Ax}^\lambda_{\neg A,A})$ $\quad \neg A, A$

$(\mathrm{Ax}^\lambda_{\forall x \in uA})$ $\quad \neg\forall x \in L_\lambda(x \in u \to A), \forall x \in uA$ if $u \in \mathcal{T}_\lambda \cup Var, u \neq x$

$(\mathrm{Ax}^\lambda_{\exists x \in uA})$ $\quad \neg\exists x \in L_\lambda(x \in u \wedge A), \exists x \in uA$ if $u \in \mathcal{T}_\lambda \cup Var, u \neq x$

$(\bigwedge_{A_0 \wedge A_1})$ $\quad \dfrac{A_0 \quad A_1}{A_0 \wedge A_1}$ $\qquad (\bigvee^k_{A_0 \vee A_1})$ $\quad \dfrac{A_k}{A_0 \vee A_1}$

$(\forall^y_{\forall x \in L_\lambda A})$ $\quad \dfrac{A_x(y)}{\forall x \in L_\lambda A}!y!$ $\qquad (\exists^v_{\exists x \in L_\lambda A})$ $\quad \dfrac{A_x(v)}{\exists x \in L_\lambda A}$

if $v \in \mathcal{T}_\lambda \cup Var$

(Cut_C) $\quad \dfrac{C \quad \neg C}{\emptyset}$

$$\mathsf{k}(\mathcal{I}) := \begin{cases} \mathsf{k}(A) \cup \mathsf{k}(i_0) & \text{if } \mathcal{I} = \exists_A^{i_0} \\ \mathsf{k}(C) & \text{if } \mathcal{I} = \mathrm{Cut}_C \\ \mathsf{k}(\Delta(\mathcal{I})) & \text{otherwise} \end{cases}$$

We denote with d, d_i RS^λ-derivations in this section and for λ we assume $\omega^\lambda = \lambda$.

DEFINITION 8.5. $\mathsf{o}(d), \mathsf{deg}(d)$
$\mathsf{o}(\mathrm{Ax}^\lambda_{\neg A, A}) := \mathsf{o}(\mathrm{Ax}^\lambda_{Qx \in uA}) := \omega^{\lambda + \mathsf{rk}_0(A)+2}$,
$\mathsf{o}(\mathcal{I} d_0 \dots d_n) := \max\{\mathsf{o}(d_0), \dots, \mathsf{o}(d_n)\} + 1$

$$\mathsf{deg}(\mathcal{I} d_0 \dots d_n) := \begin{cases} \max\{\lambda + \mathsf{rk}_0(C), \mathsf{deg}(d_0), \mathsf{deg}(d_1)\} & \text{if } \mathcal{I} = \mathrm{Cut}_C \\ \max\{0, \mathsf{deg}(d_0), \dots, \mathsf{deg}(d_n)\} & \text{otherwise} \end{cases}$$

DEFINITION 8.6. $\mathsf{FV}(d)$
$\mathsf{FV}(\mathcal{I} d_0 \dots d_n) := \mathsf{FV}(\mathcal{I}) \cup \bigcup_{i=0}^n (\mathsf{FV}(d_i) \setminus \mathsf{Eig}(\mathcal{I}))$ where

$$\mathsf{FV}(\mathcal{I}) := \begin{cases} \mathsf{FV}(\Delta(\mathcal{I})) \cup \mathsf{FV}(v) & \text{if } \mathcal{I} = \exists_A^v \\ \mathsf{FV}(\Delta(\mathcal{I})) & \text{otherwise} \end{cases}$$

and

$$\mathsf{Eig}(\mathcal{I}) := \begin{cases} \{y\} & \text{if } \mathcal{I} = \forall_A^y \\ \emptyset & \text{otherwise} \end{cases}$$

DEFINITION 8.7. A derivation d with $\mathsf{FV}(d) = \emptyset$ is called closed.

DEFINITION 8.8. $d(z/t)$ for $t \in \mathcal{T}_\lambda$

$$(\mathcal{I} d_0 \dots d_n)(z/t) := \begin{cases} \mathcal{I} d_0 \dots d_n & \text{if } \mathsf{Eig}(\mathcal{I}) = \{z\} \\ \mathcal{I}(z/t) d_0(z/t) \dots d_n(z/t) & \text{otherwise} \end{cases}$$

with

$$\mathrm{Ax}^\lambda_{\neg A, A}(z/t) := \mathrm{Ax}^\lambda_{\neg A_z(t), A_z(t)} \qquad \mathrm{Ax}^\lambda_{Qx \in uA}(z/t) := (\mathrm{Ax}^\lambda_{(Qx \in uA)_z(t)})$$
$$\textstyle\bigwedge_{A_0 \wedge A_1} := \bigwedge_{A_{0z}(t) \wedge A_{1z}(t)} \qquad \bigvee^{\mathsf{k}}_{A_0 \vee A_1}(z/t) := \bigvee^k_{A_{0z}(t) \vee A_{1z}(t)}$$
$$\mathrm{Cut}_C(z/t) := \mathrm{Cut}_{C_z(t)}$$
$$\forall_A^y(z/t) := \forall^y_{A_z(t)} \qquad \exists_A^v(z/t) := \exists^{v_z(t)}_{A_z(t)}$$

PROPOSITION 8.9. *Let d be a RS^λ-derivations and $t \in \mathcal{T}_\lambda$. Then is $d(z/t)$ a RS^λ-derivation with $\mathsf{End}(d(z/t)) \subseteq \mathsf{End}(d)_z(t)$, $\mathsf{deg}(d(z/t)) = \mathsf{deg}(d)$, $\mathsf{o}(d(z/t)) = \mathsf{o}(d)$ and $\mathsf{k}(d(z/t)) \subseteq \mathsf{k}(d) \cup \mathsf{k}(t)$.*

PROOF. Induction on the length of d. ⊣

PROPOSITION 8.10. a) $\mathsf{FV}(\mathsf{End}(d)) \subseteq \mathsf{FV}(d)$
b) $\mathsf{FV}(d(z/t)) = \mathsf{FV}(d) \setminus \{z\}$ *for* $t \in \mathcal{T}_\lambda$.

PROOF. Induction on the length of d. ⊣

Proposition 8.11. a) *For every RS^λ-derivation d exists a RS^λ-derivation d' (primitive recursive computable from d) with $\mathsf{End}(d') \subseteq \mathsf{End}(d)$, $\mathsf{deg}(d') = \mathsf{deg}(d)$, $\mathsf{o}(d') = \mathsf{o}(d)$ and $\mathsf{FV}(d') = \mathsf{FV}(\mathsf{End}(d))$.*

b) *If $d = \mathcal{I} d_0 \dots d_n$ is closed and $\mathsf{Eig}(\mathcal{I}) = \emptyset$ then $d_0 \dots d_n$ are closed.*

c) *If $d = \forall^x_A d_0$ is closed then $d_0(x/t)$ is closed for $t \in \mathcal{T}_\lambda$.*

Proof. a) Induction on the number of elements in $\mathsf{FV}(d) \setminus \mathsf{FV}(\mathsf{End}(d))$.
b) Follows from $\mathsf{FV}(d_i) \subseteq \mathsf{FV}(d) \cup \mathsf{Eig}(\mathcal{I})$.
c) Follows with the proposition above and $\mathsf{FV}(d_0) \subseteq \mathsf{FV}(d) \cup \mathsf{Eig}(\mathcal{I}) = \{x\}$. ⊣

Definition 8.1. *The notation system* $\mathbf{RS}^+$
$\mathcal{D}_0 :=$ *set of* RS^0*-derivations.*
$\mathcal{D}_\lambda :=$ *set of closed* RS^λ*-derivations for* $\lambda = \omega^\lambda$.
$\mathcal{D}_1 := \bigcup \{\mathcal{D}_\lambda : \lambda = \omega^\lambda\}$.
$\mathcal{D}^+ := \mathcal{D}_0 \cup \mathcal{D}_1$
$\mathbf{RS}^+ := (\mathcal{D}^+, \mathsf{o}, \mathsf{deg}, \mathsf{ref}, \mathsf{tp}, [])$ *with* $\mathsf{o}(d)$, $\mathsf{deg}(d)$, $\mathsf{ref}(d)$, $\mathsf{tp}(d)$, $d[i]$ *for* $d \in \mathcal{D}_0$ *defined as in the last section and* $\mathsf{o}(d)$, $\mathsf{deg}(d)$ *for* $d \in \mathcal{D}_1$ *as defined above.*
For $d \in \mathcal{D}_1$ *let* $\mathsf{ref}(d) := 0$ *and* $\mathsf{tp}(d)$, $d[i]$ *are defined as in [8] (where we replace* I *by* λ*).*

Theorem 8.12. $\mathbf{RS}^+$ *is a normal notation system for* $\mathrm{RS}(\mathcal{K})$*- derivations and is controlled by any operator which is closed under the functions* $\lambda x, y.x\#y$, $\lambda x.\omega \cdot x$, $\lambda x.\omega^x$ *and* $\lambda x.\mathsf{St}(x)$.

Proof. Induction on the length of the derivation. ⊣

Theorem 8.13. *Let the sequent* $\{\phi_1, \dots, \phi_n\}$ *be logical valid where* $\phi_1, \dots, \phi_n$ *formulas of the first order language* $(\in, (\mathrm{Ad}^\xi)_{\xi \in On})$. *Then there is an (primitive recursive computable)* RS^λ*-derivation* d *with* $\mathsf{End}(d) \subseteq \{\phi_1^\lambda, \dots, \phi_n^\lambda\}$, $\mathsf{FV}(d) = \mathsf{FV}(\mathsf{End}(d))$ *and* $\mathsf{k}(d) \subseteq \{0, \lambda\}$.

Proof. The proof is by induction on the length of a derivation of $\{\phi_1, \dots, \phi_n\}$ in an appropriate cut free calculus. For details see [8]. ⊣

Remark. For every RS^λ-derivation d there are natural numbers k, m, n with $\mathsf{o}(d) = \omega^{\lambda+n} + m$ and $\mathsf{deg}(d) \le \lambda + k$.

§9. The finitary proof system $\mathcal{D}^*$. We shall define notation systems $\mathbf{H}_\delta$ ($\delta \in \mathcal{T}(K)$) for collapsing and cut elimination. We start by introducing new inferences. See below for a motivation as well as for an explanation of the notation.

Definition 9.1.

Auxiliary inferences:

$$(\forall_w^{\beta,\alpha} F(x)) \quad \frac{\forall x^\alpha F(x)}{\forall x^\beta F(x)} \qquad \text{if } \beta \le \alpha$$

$$(\mathrm{I}_{i_0}^A) \quad \frac{A}{A_{i_0}} \qquad \text{if } A \simeq \bigwedge (A_i)_{i \in J} \text{ and } i_0 \in J$$

$$(\mathrm{S}^{\forall x^\alpha F(x)}) \quad \frac{\forall x^\alpha F(x)}{\emptyset}$$

$$(\mathrm{B}_A^{\beta,\kappa}) \quad \frac{A}{A^{\beta,\kappa}} \qquad \text{if } A \in \Sigma_1(\kappa), \beta < \kappa \in \mathit{Reg} \cup \{\mathcal{K}\}$$

Predicative cut elimination:

$$(\mathrm{R}_C) \quad \frac{C \quad \neg C}{\emptyset} \qquad \text{if } \mathsf{rk}(C) \notin \mathit{Reg}$$

$$(\mathrm{E}_\rho^\sigma) \quad \frac{\emptyset}{\emptyset} \qquad \text{if } \rho \le \sigma \text{ and } [\rho, \sigma[\cap \mathit{Reg} = \emptyset$$

Σ_3-Reflection:

$$(8.9)_A^{\xi,\pi} \qquad \frac{A}{\exists z^\pi (\mathrm{Ad}^\xi(z) \wedge A^{(z,\pi)})}$$

if $A \in \Sigma_3(\pi), \xi \in C(m(\pi), \pi) \cap m(\pi)$

$$(8.10)_{A_1 \wedge \ldots \wedge A_k}^{\xi,\pi} \qquad \frac{A_1, \ldots, A_k}{\exists z^\pi (\mathrm{Ad}^\xi(z) \wedge A_1^{(z,\pi)} \wedge \ldots \wedge A_k^{(z,\pi)})}$$

if $A_1, \ldots, A_k$ are sub formulas of $\Sigma_3(\pi)$-formulas and
$\xi \in C(m(\pi), \pi) \cap m(\pi), \xi > 0$

$$(\mathrm{N1})_{B[\vec{s}]}^{\xi,\pi}(t) \qquad \neg \mathrm{Ad}^\xi(t), \neg C[\vec{s}]^t, \exists z^\pi (\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z)$$

if $B[\vec{s}]$ is a conjunction of sub formulas of $\Sigma_3(\pi)$-formulas and
$C[\vec{s}]$ is normal form[1]of $B[\vec{s}], \xi \in C(m(\pi), \pi) \cap m(\pi)$,
$\xi > 0, t \in \mathcal{T}_\pi$

$$(\mathrm{N2})_{B[\vec{s}]}^{\xi,\pi} \qquad \forall z^\pi (\neg \mathrm{Ad}^\xi(z) \vee \neg C[\vec{s}]^z), \exists z^\pi (\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z)$$

if $B[\vec{s}]$ is a conjunction of sub formulas of $\Sigma_3(\pi)$-formulas and
$C[\vec{s}]$ is normal form of $B[\vec{s}], \xi \in C(m(\pi), \pi) \cap m(\pi), \xi > 0$

Stationary collapsing:

$$(\mathrm{H}_1 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B} \qquad \frac{\Gamma, B}{\forall v^{\pi}(\mathrm{Ad}^{\alpha}(v) \to \bigvee \Gamma^{(v,\mathcal{K})}), C^{(\pi,\mathcal{K})}}$$

if $B \in \Pi_3(\mathcal{K}), C \equiv \exists u^{\mathcal{K}}(\mathrm{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})})$
and all formulas in Γ are sub formulas of $\Pi_3(\mathcal{K})$-formulas

$$(\mathrm{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s) \qquad \frac{\Gamma, B}{\neg \mathrm{Ad}^{\alpha}(s), \bigvee \Gamma^{(s,\mathcal{K})}, C^{(\pi,\mathcal{K})}}$$

if $B \in \Pi_3(\mathcal{K}) C \equiv \exists u^{\mathcal{K}}(\mathrm{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})})$,
all formulas in Γ sub formulas of $\Pi_3(\mathcal{K})$-formulas and $s \in \mathcal{T}_\pi$

$$(10.1)^{\pi}_{\gamma,\Gamma} \qquad \frac{\Gamma}{\Gamma^{(\pi,\mathcal{K})}}$$

if all formulas in Γ are sub formulas of $\Pi_3(\mathcal{K})$-formulas

Impredicative cut elimination:

$$(\mathrm{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta} A(s) \qquad \frac{A(s)}{A(s)^{(\eta,\pi)}}$$

if $A(s) \in \Pi_2(\pi)$ and $\eta := \Psi^{\sigma}_{\pi}(\gamma + \omega^{\mu\cdot\zeta+\pi})$

$$(10.2)^{\mu,\pi,\xi}_{\gamma} \qquad \frac{\emptyset}{\emptyset}$$

$$\begin{array}{lcl}
\mathsf{k}(\forall^{\beta,\alpha}_{w} F(x)) & := & \mathsf{k}(\forall x^{\beta} F(x)) \\
\mathsf{k}(\mathrm{I}^{A}_{i_0}) & := & \mathsf{k}(A) \cup \mathsf{k}(i_0) \\
\mathsf{k}(\mathrm{S}^{\forall x^{\alpha} F(x)}) & := & \emptyset \\
\mathsf{k}(\mathrm{B}^{\beta,\kappa}_{A}) & := & \mathsf{k}(A^{(\beta,\kappa)})
\end{array}$$

$$\begin{array}{lcl}
\mathsf{k}(\mathrm{R}_C) & := & \emptyset \\
\mathsf{k}(\mathrm{E}^{\sigma}_{\rho}) & := & \{\sigma\}
\end{array}$$

$$\begin{array}{lcl}
\mathsf{k}((8.9)^{\xi,\pi}_{A}) & := & \{\xi,\pi\} \cup \mathsf{k}(A) \\
\mathsf{k}((8.10)^{\xi,\pi}_{A_1,\dots,A_k}) & := & \{\xi,\pi\} \cup \mathsf{k}(A_1) \cup \ldots \cup \mathsf{k}(A_k) \\
\mathsf{k}((\mathrm{N1})^{\xi,\pi}_{B[\vec{s}]}(t)) & := & \{\xi,\pi\} \cup \mathsf{k}(\vec{s}) \cup \mathsf{k}(t) \\
\mathsf{k}((\mathrm{N2})^{\xi,\pi}_{B[\vec{s}]}) & := & \{\xi,\pi\} \cup \mathsf{k}(\vec{s})
\end{array}$$

[1]For definition of normal form see page 25.

$$\begin{array}{lll}\mathsf{k}((\mathrm{H}_1 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}) & := & \{\gamma,\pi\}\cup\mathsf{k}(\Gamma)\cup\mathsf{k}(B)\\ \mathsf{k}((\mathrm{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s)) & := & \{\gamma,\pi\}\cup\mathsf{k}(\Gamma)\cup\mathsf{k}(B)\cup\mathsf{k}(s)\\ \mathsf{k}((10.1)^{\pi}_{\gamma,\Gamma}) & := & \{\gamma,\pi\}\cup\mathsf{k}(\Gamma)\end{array}$$

$$\begin{array}{lll}\mathsf{k}((\mathrm{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta}A(s)) & := & \{\gamma,\mu,\pi,\sigma,\eta\}\cup\mathsf{k}(A(s))\\ \mathsf{k}((10.2)^{\mu,\pi,\xi}_{\gamma}) & := & \{\gamma,\mu,\pi,\xi\}\end{array}$$

Most of these inferences are generally not valid. We will use them only under certain circumstances. The rule $(\forall^{\beta,\alpha}_{w}F(x))$ will be used to transfer a notation for a $RS(\mathcal{K})$-derivation into a notation for a $RS(\mathcal{K})$-derivation in which all inferences $\frac{\dots F(t)\dots(t\in\mathcal{T}_\alpha)}{\forall x^\alpha F(x)}$ are replaced by $\frac{\dots F(t)\dots(t\in\mathcal{T}_\beta)}{\forall x^\beta F(x)}$. The rule $\mathrm{I}^{A}_{i_0}$ gives a notation for a $RS(\mathcal{K})$-derivation in which all formulas A are replaced by A_{i_0} and all inferences $\frac{\dots A_i\dots(i\in J)}{A}$ by Rep_{i_0}. The rule $\mathrm{S}^{\forall x^\alpha F(x)}$ will only be used if $\mathsf{o}(h)<\alpha$. In this situation $\forall x^\alpha F(x)$ can not be the principal formula of an inference in the derivation. The rule $\mathrm{B}^{\beta,\kappa}_{A}$ will only be used if $\mathsf{o}(h)<\beta$. Since $\mathsf{o}(h)$ is in particular a bound for all witnesses i_0 of $\bigvee^{i_0}_{A}$-inferences, we are able to replace in the corresponding $RS(\mathcal{K})$-derivation every $\bigvee^{i_0}_{A}$-inference by $\bigvee^{i_0}_{A^{(\beta,\kappa)}}$. The rules R_C and $\mathrm{E}^{\sigma}_{\rho}$ correspond to the well known predicative cut elimination procedure. The inferences $\mathrm{B}^{\beta,\kappa}_{A}$, R_C and $\mathrm{E}^{\sigma}_{\rho}$ occur in the form above already in [8]. The inference $\mathrm{I}^{A}_{i_0}$ is a variant of a rule in [5]. The remaining rules are derived from and named after the theorems in [20] which treat the proof transformation of the associated infinite counterpart. They are first published in the authors Ph.D. thesis [17]. The task of the rule $(8.9)^{\xi,\pi}_{A}$ is to lift $\Pi_2(\pi)$-Reflection to $\Sigma_3(\pi)$-Reflection and with the rule $(8.10)^{\xi,\pi}_{A_1,\dots,A_k}$ we are able to reflect formulas the conjunction of which is equivalent to a $\Sigma_3(\pi)$-formula. At this place we need the new rules $\mathrm{Ref}^{\xi}_{\pi}A(s)$. The axioms $(\mathrm{N1})^{\xi,\pi}_{B[\vec{s}]}(t)$, $(\mathrm{N2})^{\xi,\pi}_{B[\vec{s}]}$ allow us to denote appropriate sub derivations.

The rule $(10.1)^{\pi}_{\gamma,\Gamma}$ is the key for this collapsing and cut elimination technique. We take a look on the proof transformations of the infinite counterpart to motivate this rule: Our goal is to transform a $RS(\mathcal{K})$-derivation of sub formulas of $\Pi_3(\mathcal{K})$-formulas Γ with cut rank $<\mathcal{K}+1$ into a $RS(\mathcal{K})$-derivation of $\Gamma^{(\pi,\mathcal{K})}$. Simultaneously we want to collapse the ordinal indices and lower the rank of the cuts. In the infinite case we use transfinite recursion. In the cases where the last inference was $\bigwedge_A$, $\bigvee^{i_0}_{A}$, $\mathrm{Ref}^{\xi}_{\pi}A(s)$ or Cut_A we can apply the induction hypothesis without difficulty to the sub derivations. We get the $RS(\mathcal{K})$-derivation we are looking for by applying the same rule if we ensure that in the case of an $\bigvee^{i_0}_{A}$-inference the witness i_0 stays as well below π as below the new ordinal index and that besides of $\mathcal{K}$ no ordinals above the new ordinal index occur in A in the case of a Cut_A-inference. The only problem is the case where the last inference was $\mathrm{Ref}_{\mathcal{K}}B$. We give a sketch of the essential proof transformation on the infinitary side below. For simplicity we understand Γ as a formula. Dots stand for unessential parts of the new proof tree.

$$
\cfrac{\cfrac{\cfrac{\vdots \quad \cfrac{\cfrac{d_0 : \Gamma, B}{\Gamma^{(\tau,\mathcal{K})}, B^{(\tau,\mathcal{K})}} \text{ by I.H.}}{\vdots}}{\Gamma^{(\tau,\mathcal{K})}, \exists u^\pi(\text{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})})}}{\ldots L_\tau \neq s, \Gamma^{(s,\mathcal{K})}, \exists u^\pi(\text{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})}) \ldots (\tau \in M^{\alpha_0} \cap (\mathsf{lev}(s)+1))} \bigwedge}{\cfrac{\neg\text{Ad}^{\alpha_0}(s), \Gamma^{(s,\mathcal{K})}, \exists u^\pi(\text{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})})}{\cfrac{\ldots \text{Ad}^{\alpha_0}(s) \to \Gamma^{(s,\mathcal{K})}, \exists u^\pi(\text{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})}) \ldots (s \in \mathcal{T}_\pi)}{\forall x^\pi(\text{Ad}^{\alpha_0}(x) \to \Gamma^{(x,\mathcal{K})}), \exists u^\pi(\text{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})})} \bigwedge}}
$$

The diagram has to be read as follows: Given a derivation d for $\Gamma, \exists u^{\mathcal{K}}(\mathsf{tran(u)} \wedge u \neq \emptyset \wedge \wedge B^{(u,\mathcal{K})})$ where the last inference was $\text{Ref}_{\mathcal{K}} B$. Then the subderivation d_0 ends with Γ, B. By the induction hypothesis (the derivation d_0 has a smaller ordinal than d) there is a derivation for $\Gamma^{(\tau,\mathcal{K})}, B^{(\tau,\mathcal{K})}$. With the inferences indicated we infer the endsequent of the diagram. In the finitary case the rules $(\text{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s)$ and $(\text{H}_1 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}$ allow us to jump over the first and second infinite inference in this subtree respectively. The rule $(8.10)^{\xi,\pi}_{\Gamma}$ gives us the counterpart $\Gamma^{(\pi,\mathcal{K})}, \neg\forall x^\pi(\text{Ad}^{\alpha_0}(x) \to \Gamma^{(x,\mathcal{K})})$ to get the desired result by a cut. The proof transformation has to be defined simultanously for all $\pi \in M^\alpha$ to have all premisses for the first $\bigwedge$-inference. This corresponds to have all notations for all subderivations of $(\text{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s)h_0$.
The rule $(10.2)^{\mu,\pi,\xi}_{\gamma}$ gives the actual cut elimination which again needs collapsing. In the case of a cut of rank $\mathcal{K}$ we use the rule $(10.1)^{\pi}_{\gamma,\Gamma}$. The essential reason why collapsing is possible is that we only look at derivations which end sequent consists of $\Sigma(\pi)$-formulas i.e. there is no π-fold branching in this tree. The problematic cases are here $\text{Ref}^{\xi}_{\pi} A(s)$-inferences. The essential steps to treat this cases are to invert the $\Pi_2(\pi)$-formula (inference $\text{I}^{A(s)}_t$), apply the induction hypothesis, use the fact that the ordinal index is now smaller which implies that the witness for the $\Sigma_1(\pi)$-formula is far below π which in turn allows us to replace π by a smaller η (inference $\text{B}^{(\eta,\pi)}_{A(s)_t}$). The inference $(\text{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta} A(s)$ gives the necessary notations for this transformations.

Definition 9.2 (Inductive definition of the set $\mathcal{D}^*$ of finitary derivations).

1. $\mathcal{D}^+ \subseteq \mathcal{D}^*$
2. $(\text{N1})^{\xi,\pi}_{B[\vec{s}]}(t), (\text{N2})^{\xi,\pi}_{B[\vec{s}]} \in \mathcal{D}^*$
3. If $h_0, h_1 \in \mathcal{D}^*$ then $\bigwedge_{A_0 \wedge A_1} h_0 h_1, \bigvee^{i_0}_{A} h_0, \text{Cut}_C h_0 h_1, \forall^{\beta,\alpha}_{w} F(x) h_0, \text{I}^{A}_{i_0} h_0,$ $\text{S}^{\forall x^\alpha F(x)} h_0, \text{B}^{\beta,\kappa}_{A} h_0, \text{R}_C h_0 h_1, \text{E}^{\sigma}_{\rho} h_0, (8.9)^{\xi,\pi}_{A} h_0, (8.10)^{\xi,\pi}_{A_1,\ldots,A_k} h_0,$ $(\text{H}_1 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B} h_0, (\text{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s) h_0, (10.1)^{\pi}_{\gamma,\Gamma} h_0,$ $(\text{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta} A(s) h_0, (10.2)^{\mu,\pi,\xi}_{\gamma} h_0 \in \mathcal{D}^*$

where the ordinals and formulas satisfy the side conditions in the inferences above.

§10. Inductive definition of $\mathsf{o}(h), \mathsf{deg}(h)$ and $\mathsf{ref}(h)$ for $h \in \mathcal{D}^*$. The ordinals $\mathsf{o}(h), \mathsf{deg}(h)$ and $\mathsf{ref}(h)$ are already defined for $h \in \mathcal{D}^+$.

$$\begin{array}{lcl} \mathsf{o}((\mathrm{N1})^{\xi,\pi}_{B[\vec{s}]}(t)) & := & \pi^\omega \\ \mathsf{deg}((\mathrm{N1})^{\xi,\pi}_{B[\vec{s}]}(t)) & := & \pi+\omega \\ \mathsf{ref}((\mathrm{N1})^{\xi,\pi}_{B[\vec{s}]}(t)) & := & 0 \end{array}$$

$$\begin{array}{lcl} \mathsf{o}((\mathrm{N2})^{\xi,\pi}_{B[\vec{s}]}) & := & \pi^\omega+3 \\ \mathsf{deg}((\mathrm{N2})^{\xi,\pi}_{B[\vec{s}]}) & := & \pi+\omega \\ \mathsf{ref}((\mathrm{N2})^{\xi,\pi}_{B[\vec{s}]}) & := & 0 \end{array}$$

$$\begin{array}{lcl} \mathsf{o}(\bigwedge_{A_0\wedge A_1} h_0h_1) & := & \max\{\mathsf{o}(h_0),\mathsf{o}(h_1)\}+1 \\ \mathsf{deg}(\bigwedge_{A_0\wedge A_1} h_0h_1) & := & \max\{\mathsf{deg}(h_0),\mathsf{deg}(h_1)\} \\ \mathsf{ref}(\bigwedge_{A_0\wedge A_1} h_0h_1) & := & \max\{\mathsf{ref}(h_0),\mathsf{ref}(h_1)\} \end{array}$$

$$\begin{array}{lcl} \mathsf{o}(\bigvee^{i_0}_A h_0) & := & \mathsf{o}(h_0)+1 \\ \mathsf{deg}(\bigvee^{i_0}_A h_0) & := & \mathsf{deg}(h_0) \\ \mathsf{ref}(\bigvee^{i_0}_A h_0) & := & \mathsf{ref}(h_0) \end{array}$$

$$\begin{array}{lcl} \mathsf{o}(\mathrm{Cut}_C h_0h_1) & := & \max\{\mathsf{o}(h_0),\mathsf{o}(h_1)\}+1 \\ \mathsf{deg}(\mathrm{Cut}_C h_0h_1) & := & \max\{\mathsf{rk}(C)+1,\mathsf{deg}(h_0),\mathsf{deg}(h_1)\} \\ \mathsf{ref}(\mathrm{Cut}_C h_0h_1) & := & \max\{\mathsf{ref}(h_0),\mathsf{ref}(h_1)\} \end{array}$$

$$\begin{array}{lcl} \mathsf{o}(\forall^{\beta,\alpha}_w F(x)h_0) & := & \mathsf{o}(h_0) \\ \mathsf{deg}(\forall^{\beta,\alpha}_w F(x)h_0) & := & \mathsf{deg}(h_0) \\ \mathsf{ref}(\forall^{\beta,\alpha}_w F(x)h_0) & := & \mathsf{ref}(h_0) \end{array}$$

$$\begin{array}{lcl} \mathsf{o}(\mathrm{I}^A_{i_0}h_0) & := & \mathsf{o}(h_0) \\ \mathsf{deg}(\mathrm{I}^A_{i_0}h_0) & := & \mathsf{deg}(h_0) \\ \mathsf{ref}(\mathrm{I}^A_{i_0}h_0) & := & \mathsf{ref}(h_0) \end{array}$$

$$\begin{array}{lcl} \mathsf{o}(\mathrm{S}^{\forall x^\alpha F(x)}h_0) & := & \mathsf{o}(h_0) \\ \mathsf{deg}(\mathrm{S}^{\forall x^\alpha F(x)}h_0) & := & \mathsf{deg}(h_0) \\ \mathsf{ref}(\mathrm{S}^{\forall x^\alpha F(x)}h_0) & := & \mathsf{ref}(h_0) \end{array}$$

$$\begin{array}{lcl} \mathsf{o}(\mathrm{B}^{\beta,\kappa}_A h_0) & := & \mathsf{o}(h_0) \\ \mathsf{deg}(\mathrm{B}^{\beta,\kappa}_A h_0) & := & \mathsf{deg}(h_0) \\ \mathsf{ref}(\mathrm{B}^{\beta,\kappa}_A h_0) & := & \mathsf{ref}(h_0) \end{array}$$

$$\begin{array}{lcl} \mathsf{o}(\mathrm{R}_C h_0h_1) & := & \mathsf{o}(h_0)\#\mathsf{o}(h_1) \\ \mathsf{deg}(\mathrm{R}_C h_0h_1) & := & \max\{\mathsf{rk}(C),\mathsf{deg}(h_0),\mathsf{deg}(h_1)\} \\ \mathsf{ref}(\mathrm{R}_C h_0h_1) & := & \max\{\mathsf{ref}(h_0),\mathsf{ref}(h_1)\} \end{array}$$

$$\begin{array}{lcl} \mathsf{o}(\mathrm{E}^\sigma_\rho h_0) & := & \hat{\varphi}_{\sigma\dot{-}\rho}\mathsf{o}(h_0) \\ \mathsf{deg}(\mathrm{E}^\sigma_\rho h_0) & := & \rho \\ \mathsf{ref}(\mathrm{E}^\sigma_\rho h_0) & := & \mathsf{ref}(h_0) \end{array}$$

$$
\begin{array}{lcl}
\mathsf{o}((8.9)^{\xi,\pi}_{A} h_0) & := & \pi^{\mathsf{o}(h_0)} \\
\mathsf{deg}((8.9)^{\xi,\pi}_{A} h_0) & := & \mathsf{deg}(h_0) \\
\mathsf{ref}((8.9)^{\xi,\pi}_{A} h_0) & := & \xi + 1
\end{array}
$$

$$
\begin{array}{lcl}
\mathsf{o}((8.10)^{\xi,\pi}_{A_1,\ldots,A_k} h_0) & := & \max\{\varepsilon_{\pi+1}, \mathsf{o}(h_0)+2\} \\
\mathsf{deg}((8.10)^{\xi,\pi}_{A_1,\ldots,A_k} h_0) & := & \max\{\pi+\omega, \mathsf{deg}(h_0)\} \\
\mathsf{ref}((8.10)^{\xi,\pi}_{A_1,\ldots,A_k} h_0) & := & \xi + 1
\end{array}
$$

$$
\begin{array}{lcl}
\mathsf{o}((\mathrm{H}_1 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B} h_0) & := & \Xi(\alpha+\pi) \\
\mathsf{deg}((\mathrm{H}_1 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B} h_0) & := & \Xi(\alpha+\pi) \\
\mathsf{ref}((\mathrm{H}_1 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B} h_0) & := & \max\{\alpha, \mathsf{ref}(h_0)\}
\end{array}
$$

$$
\begin{array}{lcl}
\mathsf{o}((\mathrm{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s) h_0) & := & \Xi(\alpha+\mathsf{lev}(s))+\omega \\
\mathsf{deg}((\mathrm{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s) h_0) & := & \Xi(\alpha+\pi) \\
\mathsf{ref}((\mathrm{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s) h_0) & := & \max\{\alpha, \mathsf{ref}(h_0)\}
\end{array}
$$

$$
\begin{array}{lcl}
\mathsf{o}((10.1)^{\pi}_{\gamma,\Gamma} h_0) & := & \Xi(\gamma+\mathcal{K}^{\mathsf{o}(h_0)}+\pi) \\
\mathsf{deg}((10.1)^{\pi}_{\gamma,\Gamma} h_0) & := & \Xi(\gamma+\mathcal{K}^{\mathsf{o}(h_0)}+\pi) \\
\mathsf{ref}((10.1)^{\pi}_{\gamma,\Gamma} h_0) & := & \max\{\gamma+\mathcal{K}^{\mathsf{o}(h_0)}, \mathsf{ref}(h_0)\}
\end{array}
$$

$$
\begin{array}{lcl}
\mathsf{o}((\mathrm{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta} A(s) h_0) & := & \Psi^{\sigma}_{\pi}(\gamma+\omega^{\mu\cdot\mathsf{o}(h_0)+\pi}) \\
\mathsf{deg}((\mathrm{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta} A(s) h_0) & := & \Psi^{\sigma}_{\pi}(\gamma+\omega^{\mu\cdot\mathsf{o}(h_0)+\pi}) \\
\mathsf{ref}((\mathrm{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta} A(s) h_0) & := & \max\{\gamma+\omega^{\mu\cdot(\mathsf{o}(h_0)+1)}, \mathsf{ref}(h_0)\}
\end{array}
$$

$$
\begin{array}{lcl}
\mathsf{o}((10.2)^{\mu,\pi,\xi}_{\gamma} h_0) & := & \Psi^{\xi}_{\pi}(\gamma+\omega^{\mu\cdot\mathsf{o}(h_0)}) \\
\mathsf{deg}((10.2)^{\mu,\pi,\xi}_{\gamma} h_0) & := & \Psi^{\xi}_{\pi}(\gamma+\omega^{\mu\cdot\mathsf{o}(h_0)}) \\
\mathsf{ref}((10.2)^{\mu,\pi,\xi}_{\gamma} h_0) & := & \max\{\gamma+\omega^{\mu\cdot\mathsf{o}(h_0)}, \mathsf{ref}(h_0)\}
\end{array}
$$

Remark. We have $\pi^\omega = (\omega^\pi)^\omega = \omega^{\pi\cdot\omega} = \varphi 0(\pi\cdot\omega)$. With $\beta \dot{-} \alpha$ we denote the unique ordinal ξ such that $\alpha+\xi=\beta$ for $\alpha \le \beta$.

§11. Inductive definition of $\mathsf{tp}(h)$ and $h[i]$ for $h \in \mathcal{D}^*, i \in |\mathsf{tp}(h)|$. $\mathsf{tp}(h)$ and $h[i]$ are already defined for $h \in \mathcal{D}^+$. The following definitions of $h[i]$ are repeated in the appendix in tree notation.

If $A \equiv A_1 \wedge \ldots \wedge A_n$ is a conjunction of sub formulas of $\Sigma_3(\pi)$-formulas then there is a $\mathcal{L}_{\mathrm{Ad}}$-formula $B[\vec{a}] \equiv B_1[\vec{a}] \wedge \ldots \wedge B_n[\vec{a}]$ and $\vec{s} \in \mathcal{T}_\pi^{<\omega}$ with $A \equiv B[\vec{s}]^\pi$. Let **GML** be the theory in $\mathcal{L}_{\mathrm{Ad}}$ which consists of the axioms (Ext), (Pair), (Union) and the axiom scheme Δ_0-Sep (translated to the richer language!). For $B[\vec{a}]$ we get a $\mathcal{L}_{\mathrm{Ad}}$-formula $C[\vec{a}]$ in Σ_3-form such that the equivalence of $B[\vec{a}]$ and $C[\vec{a}]$ is provable in **GML**. That means there are finitely many axioms $\phi_1, \ldots, \phi_k$ of **GML**, such that the sequent

$$\neg\phi_1, \ldots, \neg\phi_k, \neg B[\vec{a}], C[\vec{a}]$$

is derivable by purely logical means. Since we only need the axiom of pairing to code set tuples we may choose $C[\vec{a}]$ in a way that the equivalence of $C[\vec{a}]^y$ and

$B[\vec{a}]^y$ is provable in **GML** if y is a non empty transitive set which satisfies the axiom of pairing. Therefore there are further axioms $\psi_1, \dots, \psi_l$ of **GML** such that the sequence

$$\neg\psi_1, \dots, \neg\psi_l, \neg(\mathrm{tran}(y) \wedge y \neq \emptyset \wedge (Pair)^y), \neg C[\vec{a}]^y, B[\vec{a}]^y$$

is provable by purely logical means.
According to Theorem 8.13 there are RS^π-derivations d_0, d_1 of

$$\neg\phi_1^\pi, \dots, \neg\phi_k^\pi, \neg B[\vec{a}]^\pi, C[\vec{a}]^\pi$$

and

$$\neg\psi_1^\pi, \dots, \neg\psi_l^\pi, \neg(\mathrm{tran}(y) \wedge y \neq \emptyset \wedge (Pair)^y), \neg C[\vec{a}]^y, B[\vec{a}]^y$$

with $\mathsf{FV}(d_0) \subseteq \{a_0, \dots, a_m\}$ and $\mathsf{FV}(d_1) \subseteq \{a_0, \dots, a_m, y\}$. Since $s_0, \dots, s_m$, $L_\tau \in \mathcal{T}_\pi$, we know that $d_0(\vec{a}/\vec{s}), d_1(\vec{a}/\vec{s}, y/L_\tau)$ are RS^π-derivations for $\tau < \pi$. The derivations d_0, d_1 are computable from A and B respectively. We call $C[\vec{s}]$ the normal form of $B[\vec{s}]$ in this context.
We summarise the obvious finitary inferences $\bigwedge, \bigvee$ in the following and use the notations $\bigwedge^*, \bigvee^*$ respectively. We define:

$$\mathsf{tp}((\mathrm{N1})^{\xi,\pi}_{B[\vec{s}]}(t)) := \bigwedge\nolimits_{\neg\mathrm{Ad}^\xi(t)}$$

and

$$(\mathrm{N1})^{\xi,\pi}_{B[\vec{s}]}(t)[\tau] := \mathrm{Cut}_{C[\vec{s}]^\tau}(d_a, d_b) \text{ for } \tau \in M^\xi \cap \mathsf{lev}(t)$$

with

$$d_a := \mathrm{Ax}^*_{10}(L_\tau \neq t, \neg C[\vec{s}]^t, C[\vec{s}]^\tau)$$

$$d_b := \bigvee\nolimits^\tau_{F_0} \bigwedge\nolimits_{F_1} (\mathrm{Cut}_{\psi_0}(d_{b_1}, d_{b_2}), \bigvee\nolimits^\tau_{\mathrm{Ad}^\xi(L_\tau)} \mathrm{Ax}^*_3(L_\tau = L_\tau))$$

$$d_{b_1} := \bigwedge\nolimits^*_{\psi_0}((\mathrm{Ax}^*_6(\mathrm{tran}(L_\tau)), \bigvee\nolimits^0_{L_\tau \neq \emptyset} \mathrm{Ax}^*_3(L_0 = L_0)), \mathrm{Ax}^*_1(Pair^\tau))$$

$$d_{b_2} := \mathrm{Cut}_{\psi_1^\pi}(\mathrm{Ax}^*_1(\psi_1^\pi), \dots, \mathrm{Cut}_{\psi_l^\pi}(\mathrm{Ax}^*_1(\psi_l^\pi), d_1(\vec{a}/\vec{s}, y/L_\tau)) \dots)$$

$$\psi_0 :\equiv \mathrm{tran}(L_\tau) \wedge L_\tau \neq \emptyset \wedge Pair^\tau$$

$$F_0 :\equiv \exists z^\pi(\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z)$$

$$F_1 :\equiv \mathrm{Ad}^\xi(L_\tau) \wedge B[\vec{s}]^\tau$$

$$\mathsf{tp}((\mathrm{N2})^{\xi,\pi}_{B[\vec{s}]}) := \bigwedge\nolimits_{\forall z^\pi(\neg\mathrm{Ad}^\xi(z) \vee \neg C[\vec{s}]^z)}$$

$$(\mathrm{N2})^{\xi,\pi}_{B[\vec{s}]}[t] := \bigvee\nolimits^*_{\neg\mathrm{Ad}^\xi(t) \vee \neg C[\vec{s}]^t} (\mathrm{N1})^{\xi,\pi}_{B[\vec{s}]}(t)$$

for $t \in \mathcal{T}_\pi$.

For $h = \mathcal{I} h_0 \dots h_n$ with $\mathcal{I} = \bigwedge_{A_0 \wedge A_1}$, $\mathcal{I} = \bigvee^{i_0}_A$ or $\mathcal{I} = \mathrm{Cut}_C$ let $\mathsf{tp}(h) := \mathcal{I}$ and $h[i] := h_i$ for $i \in \mathsf{tp}(h)$.

For $h = \forall_w^{\beta,\alpha} F(x) h_0$ let

$$\mathsf{tp}(h) := \begin{cases} \bigwedge_{\forall x^\beta F(x)} & \text{if } \mathsf{tp}(h_0) = \bigwedge_{\forall x^\alpha F(x)} \\ \mathsf{tp}(h_0) & \text{otherwise} \end{cases}$$

$$h[i] := \forall_w^{\beta,\alpha} F(x)(h_0[i])$$

for $i \in |\mathsf{tp}(h)|$.

For $h = \mathrm{I}_{i_0}^A h_0$ let

$$\mathsf{tp}(h) := \begin{cases} \mathrm{Rep}_{i_0} & \text{if } \mathsf{tp}(h_0) = \bigwedge_A \\ \mathsf{tp}(h_0) & \text{otherwise} \end{cases}$$

$$h[i] := \mathrm{I}_{i_0}^A (h_0[i])$$

for $i \in |\mathsf{tp}(h)|$.

For $h = \mathrm{S}^{\forall x^\alpha F(x)} h_0$ let

$$\mathsf{tp}(h) := \mathsf{tp}(h_0) \text{ and } h[i] := \mathrm{S}^{\forall x^\alpha F(x)}(h_0[i])$$

for $i \in |\mathsf{tp}(h)|$.

For $h = \mathrm{B}_A^{\beta,\kappa} h_0$ let

$$\mathsf{tp}(h) := \begin{cases} \bigvee^{i_0}_{A^{(\beta,\kappa)}} & \text{if } \mathsf{tp}(h_0) = \bigvee^{i_0}_A \\ \mathsf{tp}(h_0) & \text{otherwise} \end{cases}$$

$$h[i] := \mathrm{B}_A^{\beta,\kappa}(h_0[i])$$

for $i \in |\mathsf{tp}(h)|$.

For $h = \mathrm{R}_C h_0 h_1$ let

$$\mathsf{tp}(h) := \begin{cases} \mathsf{tp}(h_0) & \text{if} \quad C \notin \Delta(\mathsf{tp}(h_0)) \\ \mathsf{tp}(h_1) & \text{if} \quad \neg C \notin \Delta(\mathsf{tp}(h_1)) \\ \mathrm{Cut}_{C_{i_0}} & \text{if} \quad C \in \Delta(\mathsf{tp}(h_0)), \neg C \in \Delta(\mathsf{tp}(h_1)) \end{cases}$$

$$h[i] := \begin{cases} \mathrm{R}_C h_0[i] h_1 & \text{if} \quad C \notin \Delta(\mathsf{tp}(h_0)) \\ \mathrm{R}_C h_0 h_1[i] & \text{if} \quad \neg C \notin \Delta(\mathsf{tp}(h_1)) \end{cases}$$

$$h[0] := \begin{cases} \mathrm{R}_C h_0[i_0] h_1 & \text{if} \quad C \in \Delta(\mathsf{tp}(h_0)), \neg C \in \Delta(\mathsf{tp}(h_1)), \mathsf{tp}(h_1) = \bigvee^{i_0}_{\neg C} \\ \mathrm{R}_C h_0[0] h_1 & \text{if} \quad C \in \Delta(\mathsf{tp}(h_0)), \neg C \in \Delta(\mathsf{tp}(h_1)), \mathsf{tp}(h_0) = \bigvee^{i_0}_C \end{cases}$$

$$h[1] := \begin{cases} \mathrm{R}_C h_0 h_1[0] & \text{if} \quad C \in \Delta(\mathsf{tp}(h_0)), \neg C \in \Delta(\mathsf{tp}(h_1)), \mathsf{tp}(h_1) = \bigvee^{i_0}_{\neg C} \\ \mathrm{R}_C h_0 h_1[i_0] & \text{if} \quad C \in \Delta(\mathsf{tp}(h_0)), \neg C \in \Delta(\mathsf{tp}(h_1)), \mathsf{tp}(h_0) = \bigvee^{i_0}_C \end{cases}$$

For $h = \mathrm{E}^\sigma_\rho h_0$ let

$$\mathsf{tp}(h) := \begin{cases} \mathrm{Rep}_0 & \text{if } \mathsf{tp}(h_0) = \mathrm{Cut}_C \text{ where } \rho \le \nu := \mathsf{rk}(C) < \sigma \\ \mathsf{tp}(h_0) & \text{otherwise} \end{cases}$$

$$h[i] := \begin{cases} \mathrm{E}^\nu_\rho \mathrm{R}_C \mathrm{E}^\sigma_\nu h_0[0] \mathrm{E}^\sigma_\nu h_0[1] & \text{if } \mathsf{tp}(h_0) = \mathrm{Cut}_C \text{ where } \rho \le \nu := \mathsf{rk}(C) < \sigma \\ \mathrm{E}^\nu_\rho h_0[i] & \text{otherwise} \end{cases}$$

for $i \in |\mathsf{tp}(h)|$.

For $h = (8.9)^{\xi,\pi}_A h_0$ let

$$\mathsf{tp}(h) := \begin{cases} \mathrm{Ref}^\xi_\pi(A_s) & \text{if } \mathsf{tp}(h_0) = \bigvee^s_A \\ \mathsf{tp}(h_0) & \text{otherwise} \end{cases}$$

$$h[i] := (8.9)^{\xi,\pi}_A (h_0[i])$$

for $i \in |\mathsf{tp}(h)|$.

For $h = (8.10)^{\xi,\pi}_{A_1,\dots,A_n} h_0$ let

$$\mathsf{tp}(h) := \mathrm{Cut}_{\exists z^\pi(\mathrm{Ad}^\xi(z) \wedge C[\vec{s}]^z)}$$

$$h[0] := \mathrm{Cut}_{B[\vec{s}]^\pi}(h_0, (8.9)^{\xi,\pi}_{C[\vec{s}]^\pi} d)$$

with

$$d := \mathrm{Cut}_{\phi_1^\pi}(\mathrm{Ax}^*_1(\phi_1^\pi), \dots, \mathrm{Cut}_{\phi_k^\pi}(\mathrm{Ax}^*_1(\phi_k^\pi), d_0(\vec{a}/\vec{s})) \dots)$$

where B is the $\mathcal{L}_{\mathrm{Ad}}$-formula and d_0 the derivation from page 25 for $A \equiv A_1 \wedge \dots \wedge A_n$.

$$h[1] := (\mathrm{N2})^{\xi,\pi}_{B[\vec{s}]}$$

For $h := (\mathrm{H}_1 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B} h_0$ let

$$\mathsf{tp}(h) := \bigwedge\nolimits_{\forall v^\pi(\mathrm{Ad}^\alpha(v) \to \bigvee \Gamma^{(v,\mathcal{K})})}$$

$$h[s] := \bigvee\nolimits^*_{\neg \mathrm{Ad}^\alpha(s) \vee \bigvee \Gamma^{(s,\mathcal{K})}} (\mathrm{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s) h_0$$

for $s \in \mathcal{T}_\pi$.

For $h := (\mathrm{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s) h_0$ let

$$\mathsf{tp}(h) := \bigwedge\nolimits_{\neg \mathrm{Ad}^\alpha(s)}$$

$$h[\tau] := \mathrm{Cut}_{\bigwedge \neg\Gamma^{(\tau,\mathcal{K})}}(d_1, \bigvee\nolimits^{\tau}_{C^{(\pi,\mathcal{K})}} \bigwedge\nolimits_G (d_2, \bigvee\nolimits^{*} (10.1)^{\tau}_{\gamma,\Gamma,B} h_0))$$

for $\tau \in M^\alpha \cap \mathsf{lev}(s)$ with

$$C :\equiv \exists u^{\mathcal{K}}(\mathrm{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})})$$

$$G :\equiv (\mathrm{tran}(L_\tau) \wedge L_\tau \neq \emptyset) \wedge B^{(\tau,\mathcal{K})}$$

$$d_1 := \mathrm{Ax}^*_{10}(L_\tau \neq s, \bigwedge \neg\Gamma^{(\tau,\mathcal{K})}, \bigvee \Gamma^{(s,\mathcal{K})})$$

$$d_2 := \bigwedge(\mathrm{Ax}^*_6(\mathrm{tran}(L_\tau)), \bigvee\nolimits^0_{L_\tau \neq \emptyset} \bigvee\nolimits^0_{L_0 \in L_\tau} \mathrm{Ax}^*_3(L_0 = L_0))$$

For $h := (10.1)^{\pi}_{\gamma,\Gamma} h_0$ let

$$\mathsf{tp}(h) := \begin{cases} \mathrm{Cut}_F & \text{if } \mathsf{tp}(h_0) = \mathrm{Ref}_{\mathcal{K}}(B) \\ \bigwedge_{A^{(\pi,\mathcal{K})}} & \text{if } \mathsf{tp}(h_0) = \bigwedge_A \\ \bigvee^{i_0}_{A^{(\pi,\mathcal{K})}} & \text{if } \mathsf{tp}(h_0) = \bigvee^{i_0}_A \\ \mathrm{Cut}_{D^{(\pi,\mathcal{K})}} & \text{if } \mathsf{tp}(h_0) = \mathrm{Cut}_D \\ \mathsf{tp}(h_0) & \text{otherwise} \end{cases}$$

with $F :\equiv \exists v^\pi(\mathrm{Ad}^{\hat{\alpha_0}}(v) \wedge \bigwedge \neg\mathsf{End}(h_0)^{(v,\mathcal{K})})$, $\hat{\alpha_0} := \gamma + \mathcal{K}^{\mathsf{o}(h_0[0])}$.
If $\mathsf{tp}(h_0) = \mathrm{Ref}_{\mathcal{K}}(B)$ let

$$h[0] := (8.10)^{\pi,\hat{\alpha_0}}_{\neg\mathsf{End}(h_0)^{(\pi,\mathcal{K})}} d_0$$

$$h[1] := (\mathrm{H}_1 10.1)^{\pi,\hat{\alpha_0}}_{\gamma,\mathsf{End}(h_0),B} h_0[0]$$

with $\mathsf{End}(h_0) = \{F_1, \ldots, F_m\}$ and

$$\begin{aligned} d_0 \quad := \quad & \bigwedge\nolimits_{\neg F_1^{(\pi,\mathcal{K})} \wedge \ldots \wedge \neg F_m^{(\pi,\mathcal{K})}} (\mathrm{Ax}^*_{10}(\neg F_1^{(\pi,\mathcal{K})}, F_1^{(\pi,\mathcal{K})}), \ldots \\ & \bigwedge\nolimits_{\neg F_{m-1}^{(\pi,\mathcal{K})} \wedge \neg F_m^{(\pi,\mathcal{K})}} (\mathrm{Ax}^*_{10}(\neg F_{m-1}^{(\pi,\mathcal{K})}, F_{m-1}^{(\pi,\mathcal{K})}), (\mathrm{Ax}^*_{10}(\neg F_m^{(\pi,\mathcal{K})}, F_m^{(\pi,\mathcal{K})}) \ldots). \end{aligned}$$

Otherwise let

$$h[i] := (10.1)^{\pi}_{\gamma^*,\mathsf{End}(h_0[i])} h_0[i]$$

for $i \in |\mathsf{tp}(h)|$ where

$$\gamma^* := \gamma_i := \gamma + \omega^{\mathcal{K}\cdot\mathsf{o}(h_0[i]) + \mathsf{lev}(i)}$$

if $\mathsf{tp}(h_0) = \bigwedge_{\forall x^{\mathcal{K}} F(x)}$ and $\gamma^* = \gamma$ otherwise.

For $h := (\mathrm{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta} A(s) h_0$ let

$$\mathsf{tp}(h) := \bigwedge\nolimits_{A(s)^{(\eta,\pi)}}$$

with $\eta := \Psi^\sigma_\pi(\gamma + \omega^{\mu\cdot\zeta+\pi})$.

$$h[t] := \mathrm{B}^{\eta,\pi}_{A(s)_t} (10.2)^{\mu,\pi,\sigma}_{\gamma_t} \mathrm{I}^{A(s)}_t h_0$$

for $t \in \mathcal{T}_\eta$ with $\gamma_t := \gamma + \omega^{\mu \cdot \zeta + \mathsf{lev}(t)}$.
Note that: $A(s)_t \equiv \exists x^\pi G(s,t,x)$ for $A(s) \equiv \forall y^\pi \exists x^\pi G(s,y,x)$.

For $h := (10.2)^{\mu,\pi,\xi}_\gamma h_0$ with $\mathsf{tp}(h_0) = \mathrm{Ref}^\sigma_\pi A(s)$ let

$$\mathsf{tp}(h) := \bigvee\nolimits^{\eta}_{\exists z^\pi (\mathrm{Ad}^\sigma(z) \wedge \exists u \in z A(u)^{(z,\pi)})}$$

with $\eta := \Psi^\sigma_\pi(\gamma + \omega^{\mu \cdot \mathsf{o}(h_0[0]) + \pi})$,

$$h[0] := \bigwedge\nolimits_{F_1} (\bigvee\nolimits^{\eta}_{\mathrm{Ad}^\sigma(L_\eta)} \mathrm{Ax}^*_3(L_\eta = L_\eta), \bigvee\nolimits^{s}_{F_2} (\mathrm{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta} A(s) h_0[0])$$

with $\zeta := \mathsf{o}(h_0[0])$ and

$$F_1 :\equiv \mathrm{Ad}^\sigma(L_\eta) \wedge \exists u \in L_\eta A(u)^{(\eta,\pi)}$$

$$F_2 :\equiv \exists u \in L_\eta A(u)^{(\eta,\pi)}.$$

For the remainder of this section let $\alpha_0 := \max\{\mathsf{o}(h_0[0]), \mathsf{o}(h_0[1])\}$.

For $h := (10.2)^{\mu,\pi,\xi}_\gamma h_0$ with $\mathsf{tp}(h_0) = \mathrm{Cut}_A$ and $\pi \le \mathsf{rk}(A)$ let $\mathsf{tp}(h) := \mathrm{Rep}_0$.

If $\mathsf{rk}(A) = \mathcal{K}$ let

$$h[0] := (10.2)^{\mu',\pi,\xi}_{\gamma'} (\mathrm{Cut}_{A^{(\kappa,\mathcal{K})}}((10.1)^\kappa_{\gamma,\Gamma,A} h_0[0], (10.1)^\kappa_{\gamma,\Gamma,\neg A} h_0[1]))$$

where $\kappa := \Xi(\gamma + \mathcal{K}^{\alpha_0})$, $\gamma' := \gamma + \omega^{\mathcal{K} \cdot \alpha_0} \cdot 2$, $\mu' := \Xi(\gamma + \mathcal{K}^{\alpha_0} + \kappa)$ and $\Gamma := \mathsf{End}(h_0)$.

If $\pi < \mathsf{rk}(A) \notin \mathit{Reg}$ let

$$h[0] := (10.2)^{\nu,\pi,\xi}_{\hat{\alpha_0}} \mathrm{E}^{\Psi^0_\tau(\hat{\alpha_0})}_{\bar{\nu}} (\mathrm{Cut}_A((10.2)^{\nu,\tau,0}_\gamma h_0[0], (10.2)^{\nu,\tau,0}_\gamma h_0[1]))$$

where $\hat{\alpha_0} := \gamma + \omega^{\mu \cdot \alpha_0}$, $\tau := \mathsf{St}(\mathsf{rk}(A))$ and $\nu := \mathsf{St}(\mathsf{rk}(A))^-$.

If $\pi \le \mathsf{rk}(A) \in \mathit{Reg}$ and $\alpha_0 < \mathsf{rk}(A) =: \tau$ let

$$h[0] := \begin{cases} (10.2)^{\mu,\pi,\xi}_\gamma S_{\neg A} h_0[1] & \text{if} \quad A \equiv \exists x^\tau F(x) \\ (10.2)^{\mu,\pi,\xi}_\gamma S_A h_0[0] & \text{if} \quad A \equiv \forall x^\tau F(x) \end{cases}$$

If $\pi \le \mathsf{rk}(A) \in \mathit{Reg}$ and $\pi = \tau \le \alpha_0$ let

$$h[0] := \mathrm{Cut}_{A^{(\Psi^0_\tau(\hat{\alpha_0}),\tau)}}(d_1, d_2)$$

with

$$d_1 := \mathrm{B}^{\Psi^0_\tau(\hat{\alpha_0}),\tau}_A (10.2)^{\mu,\tau,0}_\gamma h_0[0]$$

$$d_2 := (10.2)^{\mu,\tau,0}_{\hat{\alpha_0}} (\forall^{\Psi^0_\tau(\hat{\alpha_0}),\tau}_w F(x)) h_0[1]),$$

where without lost of generality $\neg A \equiv \forall x^\tau F(x)$.

If $\pi \leq \mathsf{rk}(A) \in Reg$ and $\pi < \tau \leq \alpha_0$ let

$$h[0] := (10.2)^{\nu,\pi,\xi}_{\gamma'} \mathrm{E}^{\Psi^0_\tau(\delta')}_{\bar{\nu}} \mathrm{Cut}_{A^{(\Psi^0_\tau(\hat{\alpha_0}),\tau)}}(d_1, d_2)$$

where d_1, d_2 as above and $\delta' := \hat{\alpha_0} + \omega^{\mu\cdot\alpha_0}$, $\gamma' := \delta' + \omega^{\mu\cdot\alpha_0}$, $\nu := \mathsf{St}(\Psi^0_\tau(\delta'))^-$. In all other cases i.e. if $\mathsf{tp}(h_0) \neq \mathrm{Ref}^\sigma_\pi A(s)$ and ($\mathsf{tp}(h_0) \neq \mathrm{Cut}_A$ or $\mathsf{rk}(A) < \pi$) let

$$\mathsf{tp}(h) := \mathsf{tp}(h_0) \text{ and } h[i] := (10.2)^{\mu,\pi,\xi}_{\gamma} h_0[i] \text{ for } i \in |\mathsf{tp}(h)|.$$

§12. Inductive definition of $\mathbf{H}_\delta$.

Definition 12.1 (The notation systems $\mathbf{H}_\delta$ for $RS(\mathcal{K})$-derivations).

1. $\mathcal{D}^+ \subseteq \mathbf{H}_\delta \subseteq \mathcal{D}^*$
2. If $h_0, h_1 \in \mathbf{H}_\delta$ then $\bigwedge_{A_0 \wedge A_1} h_0 h_1, \bigvee^{i_0}_A h_0, \mathrm{Cut}_C h_0 h_1 \in \mathbf{H}_\delta$.
3. If $h_0 \in \mathbf{H}_\delta, \beta \leq \alpha$ then $\forall^{\beta,\alpha}_w F(x) h_0 \in \mathbf{H}_\delta$.
4. If $h_0 \in \mathbf{H}_\delta, A \simeq \bigwedge(A_i)_{i\in J}, i_0 \in J$ then $\mathrm{I}^A_{i_0} h_0 \in \mathbf{H}_\delta$.
5. If $h_0 \in \mathbf{H}_\delta, \mathsf{o}(h_0) < \alpha$ then $\mathrm{S}^{\forall x^\alpha F(x)} h_0 \in \mathbf{H}_\delta$.
6. If $h_0, h_1 \in \mathbf{H}_\delta, \mathsf{rk}(C) \notin Reg$ then $\mathrm{R}_C h_0 h_1 \in \mathbf{H}_\delta$.
7. If $h_0 \in \mathbf{H}_\delta$, $\rho \leq \sigma$, $[\rho, \sigma[\cap Reg = \emptyset$, $\mathsf{deg}(h_0) \leq \sigma$ then $\mathrm{E}^\sigma_\rho h_0 \in \mathbf{H}_\delta$
8. $h_0 \in \mathbf{H}_\delta, A \in \Sigma_1(\kappa), \mathsf{o}(h_0) \leq \beta < \kappa \in Reg \cup \{\mathcal{K}\}$ then $\mathrm{B}^{\beta,\kappa}_A h_0 \in \mathbf{H}_\delta$.
9. If $t \in \mathcal{T}_\pi$ and $B[\vec{s}]$ a conjunction of sub formulas of $\Sigma_3(\pi)$-formulas then $(\mathrm{N1})^{\xi,\pi}_{B[\vec{s}]}(t), (\mathrm{N2})^{\xi,\pi}_{B[\vec{s}]} \in \mathbf{H}_\delta$.
10. If $h_0 \in \mathcal{D}^+$, $\xi \in C(m(\pi), \pi) \cap \pi$, $A \in \Sigma_3(\pi)$ then $(8.9)^{\xi,\pi}_A h_0 \in \mathbf{H}_\delta$.
11. If $h_0 \in \mathcal{D}^+$, $\xi \in C(m(\pi), \pi) \cap \pi$, $\xi > 0$ and $A_1, \dots, A_k$ sub formulas of $\Sigma_3(\pi)$-formulas, then $(8.10)^{\xi,\pi}_{A_1,\dots,A_k} h_0 \in \mathbf{H}_\delta$.
12. If $h_0 \in \mathbf{H}_\gamma$, $\hat{\alpha_0} := \gamma + \mathcal{K}^{\mathsf{o}(h_0)}$, $\pi \in M^{\hat{\alpha_0}}$, $s \in \mathcal{T}_\pi$, $\mathsf{NF}(\gamma, \mathcal{K}^{\mathsf{o}(h_0)})$, Γ sub formulas of $\Pi_3(\mathcal{K})$-formulas, $B \in \Pi_3(\mathcal{K}), C \equiv \exists u^{\mathcal{K}}(\mathrm{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})}) \in \Gamma$, $\mathsf{k}(h_0) \cup \mathsf{k}(\Gamma) \subseteq C(\gamma+1, \Xi(\gamma+1))$, $\mathsf{End}(h_0) \subseteq \Gamma, B$, $\hat{\alpha_0} + \pi \leq \delta$ then $(\mathrm{H}_2 10.1)^{\pi,\hat{\alpha_0}}_{\gamma,\Gamma,B}(s) h_0 \in \mathbf{H}_\delta$
13. If $h_0 \in \mathbf{H}_\gamma$, $\hat{\alpha_0} := \gamma + \mathcal{K}^{\mathsf{o}(h_0)}$, $\pi \in M^{\hat{\alpha_0}}$, $\mathsf{NF}(\gamma, \mathcal{K}^{\mathsf{o}(h_0)})$, Γ sub formulas of $\Pi_3(\mathcal{K})$-formulas, $B \in \Pi_3(\mathcal{K})$, $C \equiv \exists u^{\mathcal{K}}(\mathrm{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})}) \in \Gamma$, $\mathsf{k}(h_0) \cup \mathsf{k}(\Gamma) \subseteq C(\gamma+1, \Xi(\gamma+1))$, $\mathsf{End}(h_0) \subseteq \Gamma, B$, $\hat{\alpha_0} + \pi \leq \delta$ then $(\mathrm{H}_1 10.1)^{\pi,\hat{\alpha_0}}_{\gamma,\Gamma,B} h_0 \in \mathbf{H}_\delta$.
14. If $h_0 \in \mathbf{H}_\gamma$, $\hat{\alpha} := \gamma + \mathcal{K}^{\mathsf{o}(h_0)}$, $\pi \in M^{\hat{\alpha}}$, $\mathsf{NF}(\gamma, \mathcal{K}^{\mathsf{o}(h_0)})$, $\mathsf{deg}(h_0) \leq \mathcal{K}+1$, Γ sub formulas of $\Pi_3(\mathcal{K})$-formulas, $\mathsf{k}(h_0) \cup \mathsf{k}(\Gamma) \subseteq C(\gamma+1, \Xi(\gamma+1))$, $\mathsf{End}(h_0) \subseteq \Gamma$, $\hat{\alpha} + \pi \leq \delta$ then $(10.1)^\pi_{\gamma,\Gamma} h_0 \in \mathbf{H}_\delta$.
15. If $h_0 \in \mathbf{H}_\gamma$, $\hat{\alpha} := \gamma + \omega^{\mu\cdot\alpha_0}$, $\alpha_0 := \max\{\mathsf{o}(h_0)+1, \pi\}+1$, $\mu \in Card$, $\pi \leq \mu$, $\sigma \leq \gamma$, $\mathsf{NF}(\gamma, \omega^{\mu\cdot\mathsf{o}(h_0)})$, $\sigma \in C(m(\pi), \pi) \cap m(\pi)$, $\mathsf{deg}(h_0) \leq \bar{\mu}$, $\{\gamma, \pi, \sigma, \mu\} \cup \mathsf{k}(h_0) \subseteq C(\gamma+1, \Xi(\gamma+1)) \cap \bigcap\{C(\gamma+1, \Psi^0_\tau(\gamma+1)) : \pi \leq \tau \leq \mathcal{K}\}$, $\mathsf{End}(h_0) \setminus \{A(s)\} \subseteq \Sigma_1(\pi) \cup \Delta_0(\pi)$, $A(s) \in \Pi_2(\pi)$, $\mathsf{ref}(h_0) \leq \gamma$, $\zeta = \mathsf{o}(h_0)$, $\hat{\alpha} \leq \delta$ then $(\mathrm{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta} A(s) h_0 \in \mathbf{H}_\delta$
16. If $h_0 \in \mathbf{H}_\gamma$, $\hat{\alpha} := \gamma + \omega^{\mu\cdot\mathsf{o}(h_0)}$, $\mu \in Card$, $\pi \leq \mu, \xi \leq \gamma$, $\mathsf{NF}(\gamma, \omega^{\mu\cdot\mathsf{o}(h_0)})$, $\xi \in C(m(\pi), \pi) \cap m(\pi)$, $\mathsf{deg}(h_0) \leq \bar{\mu}$ $\{\gamma, \pi, \xi, \mu\} \cup \mathsf{k}(h_0) \subseteq C(\gamma+1, \Xi(\gamma+1)) \cap \bigcap\{C(\gamma+1, \Psi^0_\tau(\gamma+1)) : \pi \leq \tau \leq \mathcal{K}\}$ $\mathsf{End}(h_0) \subseteq \Sigma_1(\pi) \cup \Delta_0(\pi)$, $\mathsf{ref}(h_0) \leq \gamma$, $\hat{\alpha} \leq \delta$ then $(10.2)^{\mu,\pi,\xi}_\gamma h_0 \in \mathbf{H}_\delta$.

Only now we finished the definition of $\mathbf{H}_\delta$ and only now we have stated more precisely in which context to use the inferences given in section 9. We refer to our explanations above for a motivation of the conditions in the definition. The requirement $\mathsf{k}(h_0) \cup \mathsf{k}(\Gamma) \subseteq C(\gamma+1, \Xi(\gamma+1))$ in 12.-14. and the requirement $\{\gamma, \pi, \xi, \mu\} \cup \mathsf{k}(h_0) \subseteq C(\gamma+1, \Xi(\gamma+1)) \cap \bigcap\{C(\gamma+1, \Psi^0_\tau(\gamma+1)) : \pi \leq \tau \leq \mathcal{K}\}$ in 15. and 16. are the restrictions already mentioned necessary to keep the ordinals occurring in the witnesses below the collapsed ordinal index and to ensure that the cut formulas satisfy the induction hypothesis. The condition $\mathsf{NF}(\gamma, \mathcal{K}^{\mathsf{o}(h_0)})$ allows us to conclude $\gamma \in C(\alpha, \beta)$. That $\xi \in C(m(\pi), \pi) \cap m(\pi)$ is equivalent to M^ξ stationary in π as we have already seen. Note that the inferences $(8.9)^{\xi,\pi}_A$ and $(8.10)^{\xi,\pi}_{A_1,\dots,A_k}$ can only be applied to derivations $h_0 \in \mathcal{D}^+$.

Definition 12.2 (The operators $\mathcal{H}_\delta$).
The operators $\mathcal{H}_\delta$ are defined by

$$\mathcal{H}_\delta(X) := \bigcap\{C(\alpha, \beta) : X \subseteq C(\alpha, \beta) \wedge \delta < \alpha\}.$$

The main task of the operators is to ensure that the ordinal indexing is strongly monotone after collapsing i.e. the notations chosen here for sub derivations must have a smaller ordinal then the notation of the derivation. The concept of operator controlled derivations was first introduced in [4]. The next proposition summarises the essential properties of operators:

Proposition 12.3.

i) *$\mathcal{H}_\delta(X) \subseteq \mathcal{H}_\gamma(X)$ for $\delta < \gamma$.*
ii) *The operators $\mathcal{H}_\delta$ are closed under the functions $+, \#, \cdot, \varphi, \hat{\varphi}$ and $(\sigma \mapsto \Omega_\sigma)_{\sigma < \mathcal{K}}$.*
iii) *If $\xi, \pi, \alpha \in \mathcal{H}_\delta(X)$ and $\xi \leq \alpha \leq \delta$ then $\Psi^\xi_\pi(\alpha) \in \mathcal{H}_\delta(X)$.*
iv) *If $\Omega_\sigma \leq \eta < \Omega_{\sigma+1} < \mathcal{K}$ and $\eta \in \mathcal{H}_\delta(X)$ then $\sigma, \Omega_\sigma, \Omega_{\sigma+1} \in \mathcal{H}_\delta(X)$.*

Proof. See [20]. ⊣

The next theorem is the central statement of this paper:

Theorem 12.4. $(\mathbf{H}_\delta, \mathsf{o}, \mathsf{deg}, \mathsf{ref}, \mathsf{tp}, [])$ *is a normal notation system for $RS(\mathcal{K})$-derivations and is controlled by the operator $\mathcal{H}_\delta$.*

Proof. The proof is a rather tedious verification of the conditions defining operator controlled notation systems. There are more than fifteen main cases and several sub cases. We just show two cases to give a flavour of the argument. The essential point is here that the whole argument works by induction on the length of the derivation d. For a complete proof see [17].
Case XV. $h = (10.2)^{\mu,\pi,\xi}_\gamma h_0$
We have $h_0 \in \mathbf{H}_\gamma,\ \hat{\alpha} := \gamma + \omega^{\mu \cdot \mathsf{o}(h_0)},\ \mu \in Card,\ \pi \leq \mu,$
$\xi \leq \gamma,\ \mathsf{NF}(\gamma, \omega^{\mu \cdot \mathsf{o}(h_0)}),\ \xi \in C(m(\pi), \pi) \cap m(\pi),\ \mathsf{deg}(h_0) \leq \bar{\mu},$
$\{\gamma, \pi, \xi, \mu\} \cup \mathsf{k}(h_0) \subseteq C(\gamma+1, \Xi(\gamma+1)) \cap \bigcap\{C(\gamma+1, \Psi^0_\tau(\gamma+1)) : \pi \leq \tau \leq \mathcal{K}\},$
$\mathsf{End}(h_0) \subseteq \Sigma_1(\pi) \cup \Delta_0(\pi)$ and $\mathsf{ref}(h_0) \leq \gamma,\ \hat{\alpha} \leq \delta.$

Case XV.1. $\mathsf{tp}(h_0) = \mathrm{Ref}^{\sigma}_{\pi}(A(s))$
a)

$$\Delta(\mathsf{tp}(h)) = \{\exists z^{\pi}(\mathrm{Ad}^{\sigma}(z) \wedge \exists u \in zA(u)^{(z,\pi)})\} = \Delta(\mathsf{tp}(h_0)) \subseteq \mathsf{End}(h_0) = \mathsf{End}(h)$$

b)

$$\mathsf{End}(h[0]) = \{\mathrm{Ad}^{\sigma}(L_{\eta}) \wedge \exists u \in L_{\eta}A(u)^{\eta,\pi}\} = \Delta_0(\mathsf{tp}(h))$$

c) Since $\mathsf{tp}(h_0) = \mathrm{Ref}^{\sigma}_{\pi}(A(s))$ we get from I.H. j) $\mathsf{o}(h_0[0]) + 1 < \mathsf{o}(h_0)$, and with $\pi \leq \mu$ follows

$$\gamma + \omega^{\mu\cdot\mathsf{o}(h_0[0])+\pi} < \gamma + \omega^{\mu\cdot\mathsf{o}(h_0)}.$$

Let $d := h_0$ or $d := h_0[0]$. Then $\gamma, \pi, \xi, \mu \in C(\gamma+1, \Psi^0_{\pi}(\gamma+1)) \subseteq C(\gamma+\omega^{\mu\cdot\mathsf{o}(d)}, \pi)$ and by I.H. l),m) we get $\mathsf{o}(d) \in \mathcal{H}_{\gamma}(\mathsf{k}(h_0)) \subseteq C(\gamma + \omega^{\mu\cdot\mathsf{o}(d)}, \pi)$.
Therefore

$$\gamma + \omega^{\mu\cdot\mathsf{o}(d)(+\pi)} \in C(\gamma + \omega^{\mu\cdot\mathsf{o}(d)}, \pi).$$

Since $\xi \in C(m(\pi), \pi) \cap m(\pi)$ due to 2.17 we get

$$\Psi^{\xi}_{\pi}(\gamma + \omega^{\mu\cdot\mathsf{o}(d)}) < \pi$$

and therefore successively

$$\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)}, \pi, \gamma \in C(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)}, \Psi^{\xi}_{\pi}(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)}))$$

by the definition of the Ψ-function and $\mathsf{NF}(\gamma, \omega^{\mu\cdot\mathsf{o}(h_0)})$. Therefore

$$\Psi^0_{\pi}(\gamma + 1) \leq \Psi^{\xi}_{\pi}(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)})$$

and

$$\mathcal{H}_{\gamma}(\mathsf{k}(h_0)) \subseteq C(\gamma + 1, \Psi^0_{\pi}(\gamma + 1)) \subseteq C(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)}, \Psi^{\xi}_{\pi}(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)})).$$

Since $\mathsf{tp}(h_0) = \mathrm{Ref}^{\sigma}_{\pi}(A(s))$ it is $\sigma \in \mathsf{tp}(h_0)$ and we get

$$\gamma, \pi, \mu, \sigma, \mathsf{o}(h_0[0]) \in C(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)}, \Psi^{\xi}_{\pi}(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)}))$$

and finally

$$\Psi^{\xi}_{\pi}(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0[0])+\pi}) < \Psi^{\xi}_{\pi}(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)}).$$

Since $\mathsf{o}(h) = \Psi^{\xi}_{\pi}(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)})$ is strongly critical follows the claim.
d) is part of c).
e) By I.H.

$$\mathsf{ref}(h[0]) = \max\{\gamma+\omega^{\mu\cdot(\mathsf{o}(h_0[0])+1)}, \mathsf{ref}(h_0[0])\} \leq \max\{\gamma+\omega^{\mu\cdot\mathsf{o}(h_0)}, \mathsf{ref}(h_0)\} = \mathsf{ref}(h).$$

f) trivial.
g) As already shown in c) $\eta = \Psi^{\xi}_{\pi}(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0[0])+\pi})) < \mathsf{o}(h)$.
h)-j) trivial.
k) We have $\mathsf{k}(\mathsf{End}(h)) = \mathsf{k}(\mathsf{End}(h_0)) \subseteq \mathsf{k}(h_0) \subseteq \mathsf{k}(h)$ by I.H..
l) We have $\mathsf{k}(\mathsf{tp}(h)) = \{\eta\} \cup \mathsf{k}(\mathsf{tp}(h_0))$. By I.H. l),m) we have

$$\{\mathsf{o}(h_0), \mathsf{o}(h_0[0])\} \cup \mathsf{k}(\mathsf{tp}(h_0)) \subseteq \mathcal{H}_{\gamma}(\mathsf{k}(h_0)) \subseteq \mathcal{H}_{\delta}(\mathsf{k}(h)).$$

Since $\gamma, \mu, \pi, \xi \in \mathsf{k}(h)$ and $\sigma \in \mathsf{k}(h_0) \subseteq \mathsf{k}(h)$ follows by $\gamma + \omega^{\mu\cdot\mathsf{o}(h_0[0])+\pi} < \gamma + \omega^{\mu\cdot\mathsf{o}(h_0)} \leq \delta$ that

$$\mathsf{o}(h), \eta \in \mathcal{H}_{\delta}(\mathsf{k}(h)).$$

m) By I.H. $\mathsf{k}(h_0[0]) \subseteq \mathcal{H}_\gamma(\mathsf{k}(h_0))$ and therefore $\mathsf{k}(h[0]) \subseteq \mathcal{H}_\delta(\mathsf{k}(h))$. In l) we had already proved $\eta \in \mathcal{H}_\delta(\mathsf{k}(h))$. Therefore $\mathsf{o}(h[0]) = \omega^{\eta+1} + 1 \in \mathcal{H}_\delta(\mathsf{k}(h))$.
n) Let

$$C := C(\gamma+1, \Xi(\gamma+1)) \cap \bigcap\{C(\gamma+1, \Psi^0_\tau(\gamma+1)) : \pi \leq \tau < \mathcal{K}\}.$$

By I.H. $h_0[0] \in \mathbf{H}_\gamma$ and by assumption $\mu \in Card, \pi \leq \mu$.
Since $\mathsf{tp}(h_0) = \mathrm{Ref}^\sigma_\pi(A(s))$ we get by I.H. j) and the assumption

$$\sigma \leq \mathsf{ref}(h_0) \leq \gamma \text{ and } \sigma \in C(m(\pi), \pi) \cap m(\pi).$$

From $\mathsf{NF}(\gamma, \omega^{\mu\cdot\mathsf{o}(h_0)})$ follows $\mathsf{NF}(\gamma, \omega^{\mu\cdot\mathsf{o}(h_0[0])+\pi})$ because of $\pi \leq \mu$ together with I.H. c) and from $\mathsf{k}(h_0) \subseteq C$ follows $\mathcal{H}_\gamma(\mathsf{k}(h_0)) \subseteq C$ which implies with I.H. m)

$$\mathsf{k}(h_0[0]) \subseteq \mathcal{H}_\gamma(\mathsf{k}(h_0)) \subseteq C.$$

Since $\sigma \in \mathsf{k}(\mathsf{tp}(h_0)) \subseteq \mathcal{H}_\gamma(\mathsf{k}(h_0)) \subseteq C$ by I.H. l) follows together with the assumption $\{\gamma, \pi, \sigma, \mu\} \cup \mathsf{k}(h_0) \subseteq C$.
Further we have

$$\mathsf{End}(h_0[0]) \setminus \{A(s)\} \subseteq \mathsf{End}(h_0) \subseteq \Sigma_1(\pi) \cup \Delta_0(\pi)$$

by I.H. b), $A(s) \in \Pi_2(\pi)$ and $\mathsf{ref}(h_0[0]) \leq \mathsf{ref}(h_0) \leq \gamma$ by I.H. e) and finally $\mathsf{o}(h_0[0]) + 1, \pi < \mathsf{o}(h_0)$ since $\mathsf{tp}(h_0) = \mathrm{Ref}^\sigma_\pi(A(s))$ by I.H. j) which gives

$$\gamma + \omega^{\mu\cdot\alpha_0} \leq \gamma + \omega^{\mu\cdot\mathsf{o}(h_0)} \leq \delta$$

for $\alpha_0 := \max\{\mathsf{o}(h_0[0]) + 1, \pi\} + 1$.

Case XV.2. $\mathsf{tp}(h_0) = \mathrm{Cut}_A$ and $\mathsf{rk}(A) = \mathcal{K}$:
a)

$$\Delta(\mathsf{tp}(h)) = \Delta(\mathsf{tp}(\mathrm{Rep}_0)) = \emptyset \subseteq \mathsf{End}(h)$$

b) From $\mathsf{NF}(\gamma, \omega^{\mathcal{K}\cdot\mathsf{o}(h_0)})$ follows $\mathsf{NF}(\gamma, \mathcal{K}^{\alpha_0})$ for $\alpha_0 := \max\{\mathsf{o}(h_0[0]), \mathsf{o}(h_0[1])\}$. Therefore $\gamma + 1 \in C(\gamma + \mathcal{K}^{\alpha_0}, \Xi(\gamma + \mathcal{K}^{\alpha_0}))$ and further

$$\Xi(\gamma+1) < \Xi(\gamma + \mathcal{K}^{\alpha_0}) =: \kappa.$$

Since

$$\pi \in C(\gamma+1, \Xi(\gamma+1)) \cap \mathcal{K} \subseteq C(\gamma + \mathcal{K}^{\alpha_0}, \kappa) \cap \mathcal{K} = \kappa$$

follows

$$\mathsf{End}(h_0) = \mathsf{End}(h_0)^{(\kappa,\mathcal{K})}$$

from $\mathsf{End}(h_0) \subseteq \Sigma_1(\pi) \cup \Delta_0(\pi)$.
We get

$$\mathsf{End}(h[0]) = \mathsf{End}(h_0) = \mathsf{End}(h).$$

c) Let

$$C := C(\gamma+1, \Xi(\gamma+1)) \cap \bigcap\{C(\gamma+1, \Psi^0_\tau(\gamma+1)) : \pi \leq \tau \leq \mathcal{K}\}.$$

We have $\mathsf{o}(h[0]) = \Psi^\xi_\pi(\gamma' + \omega^{\mu'\cdot(\mu'+1)})$ where $\gamma' := \gamma + \omega^{\mathcal{K}\cdot\alpha_0} \cdot 2$, $\mu' := \Xi(\hat{\alpha}_0 + \kappa)$ and $\mathsf{o}(h) = \Psi^\xi_\pi(\gamma + \omega^{\mu\cdot\mathsf{o}(h_0)})$.

Let $\eta := \gamma' + \omega^{\mu' \cdot (\mu'+1)}$.
By assumption and I.H. m) we have

$$\mathsf{o}(h_0[0]), \mathsf{o}(h_0[1]) \in \mathcal{H}_\gamma(\mathsf{k}(h_0)) \subseteq C.$$

Therefore $\alpha_0 \in C$ and since by assumption $\gamma \in C$ follows $\gamma + \mathcal{K}^{\alpha_0} \in C$. By I.H. l) follows $\gamma + \mathcal{K}^{\mathsf{o}(h_0)} \in C$ and because of $C \subseteq C(\gamma+1, \pi)$

$$\Psi_\pi^\xi(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}) < \pi,$$

since M^ξ stationary in π because of $\xi \in C(m(\pi), \pi) \cap m(\pi)$. Since $\kappa = \Xi(\gamma + \mathcal{K}^{\alpha_0}) \in C(\gamma + \mathcal{K}^{\alpha_0} \cdot 2 + \omega^{\mu' \cdot (\mu'+1)}, \pi)$ follows further

$$\Psi_\pi^\xi(\gamma' + \omega^{\mu' \cdot (\mu'+1))}) < \pi.$$

Since $\gamma + 1 \in C(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}, \Psi_\pi^\xi(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}))$ follows

$$C \subseteq C(\gamma+1, \Psi_\pi^0(\gamma+1)) \subseteq C(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}, \Psi_\pi^\xi(\gamma + \mathcal{K}^{\mathsf{o}(h_0)})).$$

Therefore $\gamma + \mathcal{K}^{\alpha_0} \in C(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}, \Psi_\pi^\xi(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}))$. Since by I.H. $\alpha_0 < \mathsf{o}(h_0)$ follows $\kappa \in C(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}, \Psi_\pi^\xi(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}))$. It follows $\gamma + \mathcal{K}^{\alpha_0} + \kappa \in C(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}, \Psi_\pi^\xi(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}))$ and because of $\gamma + \mathcal{K}^{\alpha_0} + \kappa < \gamma + \mathcal{K}^{\mathsf{o}(h_0)}$ further $\mu' \in C(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}, \Psi_\pi^\xi(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}))$. It follows $\gamma' + \omega^{\mu' \cdot (\mu'+1)} \in C(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}, \Psi_\pi^\xi(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}))$ and because of $\pi, \xi \in C$ and $\gamma' + \omega^{\mu' \cdot (\mu'+1)} < \gamma + \mathcal{K}^{\mathsf{o}(h_0)}$

$$\mathsf{o}(h[0]) = \Psi_\pi^\xi(\gamma' + \omega^{\mu' \cdot (\mu'+1)}) < \Psi_\pi^\xi(\gamma + \mathcal{K}^{\mathsf{o}(h_0)}) = \mathsf{o}(h).$$

d)

$$\mathsf{deg}(h[0]) = \mathsf{o}(h[0]) < \mathsf{o}(h) = \mathsf{deg}(h)$$

e) We have $\mu = \mathcal{K}$ and therefore

$$\begin{aligned}\mathsf{ref}(h[0]) &= \max\{\gamma + \mathcal{K}^{\mathsf{o}(h_0[0])}, \gamma + \mathcal{K}^{\mathsf{o}(h_0[1])}, \gamma' + \omega^{\mu' \cdot (\mu'+1)}, \mathsf{ref}(h_0[0]), \mathsf{ref}(h_0[1])\} \\ &\leq \max\{\gamma + \omega^{\mu \cdot \mathsf{o}(h_0)}, \mathsf{ref}(h_0)\} \\ &= \mathsf{ref}(h),\end{aligned}$$

since $\mu' < \mathcal{K}$ and by I.H. c) $\gamma' < \gamma + \mathcal{K}^{\mathsf{o}(h_0)} = \gamma + \omega^{\mathcal{K} \cdot \mathsf{o}(h_0)}$.
f)-j) trivial.
k) follows from the I.H..
l) We have $\mathsf{k}(\mathsf{tp}(h)) = \emptyset$ and $\mathsf{o}(h) \in \mathcal{H}_\delta(\mathsf{k}(h))$ was proved already above.
m) We have $\mathsf{k}(h[0]) = \{\mu', \pi, \xi, \gamma', \gamma, \kappa\} \cup \mathsf{k}(A^{(\kappa, \mathcal{K})}) \cup \mathsf{k}(\mathsf{End}(h_0))$. Since $\mathsf{tp}(h_0) = \mathrm{Cut}_A$ follows by I.H. l) $\mathsf{k}(A) \subseteq \mathcal{H}_\gamma(\mathsf{k}(h_0)) \subseteq \mathcal{H}_\delta(\mathsf{k}(h))$. Further we have $\mathsf{k}(\mathsf{End}(h_0)) \subseteq \mathcal{H}_\delta(\mathsf{k}(h))$ by I.H. k). We have $\pi, \xi, \gamma \in \mathsf{k}(h)$ and by I.H. $\alpha_0 \in \mathcal{H}_\delta(\mathsf{k}(h))$ and therefore $\gamma + \mathcal{K}^{\alpha_0} \in \mathcal{H}_\delta(\mathsf{k}(h))$. Since $\gamma + \mathcal{K}^{\alpha_0} < \gamma + \mathcal{K}^{\mathsf{o}(h_0)} \leq \delta$ follows

$$\kappa = \Xi(\gamma + \mathcal{K}^{\alpha_0}) \in \mathcal{H}_\delta(\mathsf{k}(h))$$

and therefore $\gamma + \mathcal{K}^{\alpha_0} + \kappa \in \mathcal{H}_\delta(\mathsf{k}(h))$ and because of $\gamma + \mathcal{K}^{\alpha_0} + \kappa < \gamma + \mathcal{K}^{\mathsf{o}(h_0)} \leq \delta$

$$\mu' = \Xi(\gamma + \mathcal{K}^{\alpha_0} + \kappa) \in \mathcal{H}_\delta(\mathsf{k}(h)).$$

Since we have $\gamma' = \gamma + \mathcal{K}^{\alpha_0} \cdot 2 \in \mathcal{H}_\delta(\mathsf{k}(h))$ follows $\gamma' + \omega^{\mu' \cdot (\mu'+1)} \in \mathcal{H}_\delta(\mathsf{k}(h))$ and because of $\xi, \pi \in \mathcal{H}_\delta(\mathsf{k}(h))$ and $\gamma' + \omega^{\mu' \cdot (\mu'+1)} < \delta$

$$\mathsf{o}(h[0]) = \Psi_\pi^\xi(\gamma' + \omega^{\mu' \cdot (\mu'+1)}) \in \mathcal{H}_\delta(\mathsf{k}(h)).$$

n) By I.H. $h_0[0], h_0[1] \in \mathcal{H}_\gamma$. Since $\pi < \kappa$ and by I.H. m) $\mathsf{o}(h_0[0]), \mathsf{o}(h_0[1]) \in \mathcal{H}_\gamma(\mathsf{k}(h_0)) \subseteq C(\gamma{+}1, \pi)$ follows $\alpha_1 \in C(\gamma{+}\mathcal{K}^{\alpha_0}, \kappa)$ for $\alpha_1 := \min\{\mathsf{o}(h_0[0]), \mathsf{o}(h_0[1])\}$. Since $\kappa \in M^{\gamma+\mathcal{K}^{\alpha_0}}$ follows $\kappa \in M^{\gamma+\mathcal{K}^{\alpha_1}}$ i.e. $\kappa \in M^{\gamma+\mathcal{K}^{\mathsf{o}(h_0[i])}}$ for $i = 0, 1$.
From $\mathsf{NF}(\gamma, \mathcal{K}^{\mathsf{o}(h_0)})$ and I.H. c) follows $\mathsf{NF}(\gamma, \mathcal{K}^{\mathsf{o}(h_0[i])})$ and we have $\mathsf{deg}(h_0[i]) \leq \mathsf{deg}(h_0) \leq \bar{\mu} = \mathcal{K} + 1$ by I.H. d) for $i = 0, 1$.
Since $\mathsf{rk}(A) = \mathcal{K}$ and $\mathsf{End}(h_0) \subseteq \Sigma_1(\pi) \cup \Delta_0(\pi)$ follows $\mathsf{End}(h_0), (\neg)A$ are sub formulas of $\Pi_3(\mathcal{K})$-formulas.
By I.V. k),l),m) follows

$$\begin{aligned}\mathsf{k}(h_0[i]) \cup \mathsf{k}(\mathsf{End}(h_0)) \cup \mathsf{k}(A) &\subseteq \mathsf{k}(h_0[i]) \cup \mathsf{k}(h_0) \cup \mathsf{k}(\mathsf{tp}(h_0)) \\ &\subseteq \mathcal{H}_\delta(\mathsf{k}(h_0)) \subseteq C(\gamma+1, \Xi(\gamma+1))\end{aligned}$$

Since $\gamma + \mathcal{K}^{\alpha_i} + \kappa < \gamma + \mathcal{K}^{\alpha_0} \cdot 2$ follows

$$(10.1)^\kappa_{\gamma,\Gamma,A} h_0[0], (10.1)^\kappa_{\gamma,\Gamma,\neg A} h_0[1] \in \mathbf{H}_{\gamma'}.$$

We have $\mu' \in Card$ and since $\kappa \in C(\hat{\alpha_0} + \kappa, \Xi(\hat{\alpha_0} + \kappa))$ and $\pi < \kappa$ as already shown in b) follows

$$\pi \in C(\hat{\alpha_0} + \kappa, \Xi(\hat{\alpha_0} + \kappa)) \cap \mathcal{K} = \Xi(\hat{\alpha_0} + \kappa) = \mu'.$$

By assumption we have $\xi \leq \gamma < \gamma'$ and $\xi \in C(m(\pi), \pi) \cap m(\pi)$ and $\mathsf{NF}(\gamma', \omega^{\mu' \cdot (\mu'+1)})$ because of $\mu' < \mathcal{K}$.
Further we have

$$\mathsf{deg}(\underbrace{\mathrm{Cut}_{A^{(\kappa,\mathcal{K})}}(10.1^\kappa_{\gamma,\Gamma,A} h_0[0], 10.1^\kappa_{\gamma,\Gamma,\neg A} h_0[1])}_{=:h_1}) = \mu'.$$

Let $C' := C(\gamma' + 1, \Xi(\gamma' + 1)) \cap \bigcap\{C(\gamma' + 1, \Psi^0_\tau(\gamma' + 1)) : \pi \leq \tau \leq \mathcal{K}\}$.
Since $\gamma \in C(\gamma'+1, \Xi(\gamma'+1))$ because of $\mathsf{NF}(\gamma, \mathcal{K}^{\alpha_0})$ follows $C(\gamma+1, \Xi(\gamma+1)) \subseteq C(\gamma' + 1, \Xi(\gamma' + 1))$.
If $\Psi^0_\tau(\gamma' + 1) = \tau$ then $\Psi^0_\tau(\gamma + 1) \leq \Psi^0_\tau(\gamma' + 1)$ and therefore

$$C(\gamma + 1, \Psi^0_\tau(\gamma + 1)) \subseteq C(\gamma' + 1, \Psi^0_\tau(\gamma' + 1)).$$

If $\Psi^0_\tau(\gamma' + 1) < \tau$ then $\gamma', \tau \in C(\gamma' + 1, \Psi^0_\tau(\gamma' + 1))$ and therefore we have $\gamma \in C(\gamma'{+}1, \Psi^0_\tau(\gamma'{+}1))$ because of $\mathsf{NF}(\gamma, \mathcal{K}^{\alpha_0})$. It follows $\Psi^0_\tau(\gamma{+}1) < \Psi^0_\tau(\gamma'{+}1)$ and therefore in this case as well

$$C(\gamma + 1, \Psi^0_\tau(\gamma + 1)) \subseteq C(\gamma' + 1, \Psi^0_\tau(\gamma' + 1)).$$

Therefore $C \subseteq C'$. From $\gamma \in C$ and $\mathsf{o}(h_0[0]), \mathsf{o}(h_0[1]) \in \mathcal{H}_\gamma(\mathsf{k}(h_0)) \subseteq C$ follows therefore $\gamma{+}\mathcal{K}^{\alpha_0} \in C'$ and since $\gamma{+}\mathcal{K}^{\alpha_0} < \gamma'$ further $\kappa \in C'$ and as well $\mu' \in C'$ because of $\gamma + \mathcal{K}^{\alpha_0} + \kappa < \gamma'$.
Altogether we have

$$\{\gamma', \pi, \xi, \mu'\} \cup \mathsf{k}(h_1) = \{\gamma', \pi, \xi, \mu', \gamma\} \cup \mathsf{k}(h_0[0]) \cup \mathsf{k}(h_0[1]) \cup \mathsf{k}(\Gamma, A) \subseteq C'.$$

Since $\mathsf{End}(h_1) = \mathsf{End}(h_0) \subseteq \Sigma_1(\pi) \cup \Delta_0(\pi)$ and

$$\mathsf{ref}(h_1) = \max\{\gamma + \mathcal{K}^{\alpha_0}, \mathsf{ref}(h_0[0]), \mathsf{ref}(h_0[1])\} < \gamma'$$

because of $\mathsf{ref}(h_0[i]) \leq \mathsf{ref}(h_0) \leq \gamma$ for $i = 0, 1$ as well as $\gamma' + \omega^{\mu' \cdot (\mu'+1)} < \gamma + \omega^{\mathcal{K} \cdot \mathsf{o}(h_0)} \leq \delta$ follows $h[0] \in \mathbf{H}_\delta$. ⊣

We can proof a theorem similar to Theorem 5.3 in [8]. Let "z=HF" denote the formula $\mathsf{tran}(z) \wedge \exists x \in z(x \subseteq x) \wedge (Pair)^z \wedge (Union)^z \wedge ((x \in z_1 \vee x = z_2) - Sep)^z \wedge \forall x \in z \exists u \in z(\exists y \in u(x \in y) \wedge \mathcal{A}(u))$.

THEOREM 12.5. *Let* Π_3-Refl $\vdash \forall z($"z=HF" $\to \phi^z)$ *where* $\mathsf{FV}(\phi) = \emptyset$. *Then there is a* $\delta < \varepsilon_{\mathcal{K}+1}$ *and a* $h \in \mathbf{H}_\delta$ *with* $\mathsf{o}(h) < \Psi^0_\Omega(\varepsilon_{\mathcal{K}+1})$, $\mathsf{deg}(h) = 0$ *and* $\mathsf{End}(h) \subseteq \{\phi^{L_\omega}\}$.

PROOF. Let Π_3-Refl $\vdash \forall z($"z=HF" $\to \phi^z)$. Then there is a Conjunction χ of Axioms of Π_3-Refl such that the sequent

$$\neg(\chi \wedge \text{"z=HF"}), \phi^z$$

is derivable by purely logical means.
According to theorem 8.13 there is a $RS^{\mathcal{K}}$-derivation h_0 and $n, m < \omega$ with

$$\mathsf{End}(h_0) \subseteq \{\neg(\chi \wedge \text{"z=HF"}), \phi^z\}$$

$\mathsf{FV}(h_0) = \{z\}, \mathsf{k}(h_0) \subseteq \{0, \mathcal{K}\}, \mathsf{o}(h_0) \leq \omega^{\mathcal{K}+n} + m, \mathsf{deg}(h_0) < \mathcal{K} + \omega$.
By proposition 8.9 $h_1 := h_0(z/L_\omega)$ is a closed $RS^{\mathcal{K}}$-derivation (i.e. $h_1 \in \mathcal{D}^+$) with

$$\mathsf{End}(h_1) \subseteq \{\neg(\chi \wedge \text{"}L_\omega\text{=HF"}), \phi^{L_\omega}\}$$

$\mathsf{k}(h_1) \subseteq \{0, \mathcal{K}\}, \mathsf{o}(h_0) \leq \omega^{\mathcal{K}+n} + m, \mathsf{deg}(h_0) < \mathcal{K} + \omega$.
Further there is a RS^0-derivation h_1' with

$$\mathsf{End}(h_1') \subseteq \{\chi^{\mathcal{K}} \wedge \text{"}L_\omega\text{=HF"}\}$$

$\mathsf{o}(h_1') < \omega^{\mathcal{K}+\omega}, \mathsf{deg}(h_1') \leq \mathcal{K}$ and $\mathsf{k}(h_1') \subseteq \{0, \omega, \mathcal{K}\}$ (for $\chi = \chi_1 \wedge \ldots \wedge \chi_l$ h_1' is built by $\bigwedge$-inferences and the RS^0-derivations $\mathrm{Ax}^*_1(Pair)^\omega$, $\mathrm{Ax}^*_1(Union)^\omega$, $\mathrm{Ax}^*_6(\mathrm{tran}(L_\omega))$, $\mathrm{Ax}^*_{15}(\forall x \in L_\omega \exists u \in L_\omega(\exists y \in u(x \in y) \wedge \mathcal{A}(u))$, $\mathrm{Ax}^*_1((x \in z_1 \vee x = z_2) - Sep)^\omega$ as well as $\mathrm{Ax}^*_j \chi_i^{\mathcal{K}}$ for $i = 1, \ldots, l$ and $j = 1$ or $j = 2$ according to the kind of χ_i).
We have $\mathsf{deg}(\underbrace{\chi^{\mathcal{K}} \wedge \text{"}L_\omega\text{=HF"}}_{=:C}) < \mathcal{K} + \omega$.
Let $\mathcal{K} + k := \min\{\mathsf{deg}(C), \mathsf{deg}(h_1), \mathsf{deg}(h_1'), \mathcal{K} + 1\}$ and

$$h := \mathrm{E}_0^{\Psi^0_{\Omega_1}(\mathcal{K}^\alpha)}(10.2)_0^{\mathcal{K},\Omega_1,0}\mathrm{E}^{\mathcal{K}+k}_{\mathcal{K}+1}\mathrm{Cut}_C(h_1', h_1)$$

where $\alpha := \omega_{k \dot{-} 1}(\max\{\mathsf{o}(h_1) + 1, \mathsf{o}(h_1') + 1\})$.
Since $h_1, h_1' \in \mathcal{D}^+$ follows $h_1, h_1' \in \mathbf{H}_0$ and therefore $h_2 := \mathrm{Cut}_C(h_1', h_1) \in \mathbf{H}_0$.
We have $\mathsf{deg}(h_2) \leq \mathcal{K} + k$ and therefore $h_3 := \mathrm{E}^{\mathcal{K}+k}_{\mathcal{K}+1} h_2 \in \mathbf{H}_0$.
It is easy to verify the conditions for

$$h_4 := (10.2)_0^{\mathcal{K},\Omega_1,0} h_3 \in \mathbf{H}_{\omega^{\mathcal{K}\cdot\alpha}}$$

and therefore we have $h = \mathrm{E}_0^{\Psi^0_{\Omega_1}(\mathcal{K}^\alpha)} h_4 \in \mathbf{H}_{\omega^{\mathcal{K}\cdot\alpha}}$. ⊣

§13. A bound for the Π^0_2-Skolem functions of Π_3-Reflection.

As a first application of the notation systems $\mathbf{H}_\delta$ we are going to define a 2-ary recursive function f with $\models \forall x \in L_n \exists y \in L_{f(h,n)} \phi(x, y)$ for $h \in \mathbf{H}_\delta$ with $\mathsf{End}(h) \subseteq \{\forall x \in L_\omega \exists y \in L_\omega \phi(x, y)\}$, $\mathsf{deg}(h) = 0$, $\phi(x, y) \in \Delta_0$. We use the symbol $\models$ in this section for validity in the structure of the hereditary finite sets. By Theorem 12.5 we get from Π_3-Refl $\vdash \forall z($"z=HF" $\to \forall x \in z \forall y \in z \phi(x, y))$, $\phi(x, y) \in \Delta_0$ that

$\models \forall x \in L_n \exists y \in L_{f(h,n)} \phi(x,y)$ for an appropriate $h \in \mathbf{H}_\delta$. Since f will be defined by $<$-recursion in the sense of Takeuti [28], the recursive enumerable subsets in the structure of the hereditary finite sets are exactly the Σ_1-definable subsets of the natural numbers (Barwise [3]) and a partial function is recursive if and only if the graph of the function is recursive enumerable (e.g. Rogers [15]) we may interprete this as a characterisation of the provably recursive (provably total) functions of $\mathbf{KP} + \Pi_3$-Refl. In this section only we use transfinite induction. Since we proceed again in a way similar to [8] we just state the definitions, propositions and theorems and point out the minor differences.

DEFINITION 13.1. $2_0 := 0, 2_{m+1} := 2^{2_m}$

$$s_n := \begin{cases} L_0 & \text{if } n = 0 \\ [x \in L_{l+1} : x = s_{n_0} \vee \ldots \vee x = s_{n_k}] & \text{if } n = 2^{n_0} + \ldots + 2^{n_k}, \\ & n_0 > \ldots > n_k, l := \mathsf{lev}(s_{n_0}) \end{cases}$$

PROPOSITION 13.2. a) *s_n is an RS-term with* $\mathsf{lev}(s_n) < \omega$,
b) $\mathsf{lev}(s_n) < m$ *iff* $n < 2_m$.

PROOF. See [8]. ⊣

DEFINITION 13.3. $\mathcal{T}^*_m := \{s_n : \mathsf{lev}(s_n) < m\} = \{s_n : n < 2_m\}$.

REMARK. Note that $\mathcal{T}^*_m$ is a finite set in contrast to $\mathcal{T}_m$.

DEFINITION 13.4. $\models A :\text{iff} \begin{cases} \forall i \in J \models A_i & \text{if} \quad A \equiv \bigwedge(A_i)_{i\in J} \\ \exists i \in J \models A_i & \text{if} \quad A \equiv \bigvee(A_i)_{i\in J} \end{cases}$.
$\models \Gamma$:iff $\exists A \in \Gamma \models A$.

PROPOSITION 13.5. a) $\models \neg A$ *iff* $\not\models A$.
b) $\models s \neq t, \neg A(s), A(t)$.

PROOF. See [8]. ⊣

PROPOSITION 13.6. *For $a \in \mathcal{T}_\omega$ there is an $n < \omega$ with*

$$\models a = s_n \textit{ and } \mathsf{lev}(s_n) \leq \mathsf{lev}(a).$$

PROOF. See [8]. ⊣

PROPOSITION 13.7. *For $A \equiv \bigwedge(A_i)_{i\in\mathcal{T}_m}$ we have*

$$\models A \quad \textit{iff} \quad \models A_i \textit{ for all } i \in \mathcal{T}^*_m.$$

PROOF. See [8]. ⊣

DEFINITION 13.8. The class of $<$-recursive functions is the smallest class of arithmetical functions which contains the constant zero function, the projections and the successor function and is closed under superposition, primitive recursion and $<$-recursion, i.e. if h, g, θ are $<$-recursive then so is f where f is given by

$$f(\vec{x}, y) := \begin{cases} h(\vec{x}, y, f(\vec{x}, \theta(\vec{x}, y))) & \text{if } \theta(\vec{x}, y) < y \\ g(\vec{x}, y) & \text{otherwise.} \end{cases}$$

REMARK. $<$ denotes the ordering on $\mathcal{T}(K)$.

DEFINITION 13.9. of $f(h, n)$ for $h \in \mathbf{H}_\delta$ and $n \in \omega$

For $A \simeq \bigwedge(A_i)_{i\in J}$ let $\mid A \mid^n := \begin{cases} \mathcal{T}_m^* & \text{if } J = \mathcal{T}_m \\ \mathcal{T}_n^* & \text{if } J = \mathcal{T}_\omega \\ J & \text{if } J = \{0,1\} \\ \emptyset & \text{otherwise} \end{cases}$

$$f(h,n) := \begin{cases} f(h[i_0], n) & \text{if } \mathsf{tp}(h) = \mathrm{Rep}_{i_0} \\ \max\{f(h[0], n), \mathsf{lev}(i_0) + 1\} & \text{if } \mathsf{tp}(h) = \bigvee_A^{i_0}, \mathsf{lev}(i_0) < \omega \\ \max\{f(h[i], n) : i \in \mid A \mid^n\} & \text{if } \mathsf{tp}(h) = \bigwedge_A \\ 0 & \text{otherwise} \end{cases}$$

DEFINITION 13.10. Let $A^{n,k}$ the RS-formula which we get by replacing in A every bounded quantifier of the shape $\forall x \in L_\omega$ by $\forall x \in L_n$ and every bounded quantifier of the form $\exists x \in L_\omega$ by $\exists x \in L_k$. Let $\Gamma^{n,k} := \{A^{n,k} : A \in \Gamma\}$

PROPOSITION 13.11. *If $h \in \mathbf{H}_\delta$, $\mathsf{End}(h) \subseteq RS_\omega, \mathsf{deg}(h) = 0$ and $f(h,n) \leq k$ then $\models \mathsf{End}(h)^{n,k}$.*

PROOF. Transfinite induction on $\mathsf{o}(h)$.
Let $h \in \mathbf{H}_\delta, \mathsf{End}(h) \subseteq RS_\omega, \mathsf{deg}(h) = 0$ and $f(h,n) \leq k$.
Since $\mathsf{End}(h) \subseteq RS_\omega$ we have $\mathsf{tp}(h) \neq \mathrm{Ref}_{\mathcal{K}}A$ and $\mathsf{tp}(h) \neq \mathrm{Ref}_\pi^\xi(A(s))$. Therefore we can conclude as in [8]. ⊣

THEOREM 13.12. *If $\mathbf{KP} + \Pi_3$-Refl $\vdash \forall z(\text{“z=HF”} \to \forall x \in z \forall y \in z\phi(x,y))$ where $\phi(x,y) \in \Delta_0$ then there is a $h \in \mathbf{H}_\delta$ with $\mathsf{o}(h) < \Psi_\Omega^0(\epsilon_{\mathcal{K}+1})$ and for every natural number n we have $\models \forall x \in L_n \exists y \in L_{f(h,n)}\phi(x,y)$.*

PROOF. Follows by 12.5 and the proposition above. ⊣

§14. A conservation result. It is well known that we may interprete properties of the hereditary finite sets as arithmetical properties. We are going to prove that for $\phi \in \Delta_0$ from $\mathbf{KP} + \Pi_3$-Refl $\vdash \forall z(\text{“z=HF”} \to \forall x \in z \exists y \in z\phi(x,y))$ follows $\mathbf{PRA} + \mathrm{PRWO}(<) \vdash \forall x \exists y \mathcal{U}(\phi(x,y))$ where $\mathcal{U}(\phi)$ is a translation of the set theoretical formula ϕ in an arithmetical formula. By PRWO(<) we denote the property of $<$ that there is no infinite descending recursive function. We may axiomatise PRWO(<) by the formulas $\exists n f(n+1) \not< f(n)$ where f runs over all primitive recursive functions and may contain further parameters. Note that there are primitive recursive well-orderings which are not well-orderings (e.g. Troelstra/Schwichtenberg [30] pp. 279-284). The theory **PRA** is formulated in a first order language. The function symbols and axioms are build analogous to the primitive recursive functions and there defining equations. The symbol = is the only relation symbol and the symbol 0 the only constant symbol of the language. In **PRA** natural induction is restricted to formulas without quantifiers i.e. Δ_0-formulas of the language. Skolem introduced **PRA** 1923 [27] as an informal system (without quantifiers). The theory is discussed in length in Hilbert/Bernays [12] and serves as an example for finitary reasoning. The quantifier-free part of the theory has a lot of interesting properties. In particular it is independent of the logic based on: we can proof the same sentences from the axioms by intuitionistic as by classical logic. See as well Troelstra/van Dalen

[31] for the relevance of **PRA**.
The proof of the result announced above can be sketched as follows: If we have $\mathbf{KP} + \Pi_3\text{-Refl} \vdash \forall z(\text{“z=HF”} \rightarrow \forall x \in z \exists y \in z\phi(x,y))$ then we get a notation h of an infinitary cut-free derivation with $\mathsf{End}(h) \subseteq \{\forall x \in L_\omega \exists y \in L_\omega \phi(x,y)\}$. For every natural number n we get a notation $h(n) := \mathrm{I}_{s_n}^{\forall x \in L_\omega \exists y \in L_\omega \phi(x,y)} h$ for an infinitary derivation with $\mathsf{End}(h(n)) \subseteq \{\exists y \in L_\omega \phi(s_n, y)\}$. If we assume that the end formula of $h(n)$ is wrong then one of the premises of the last inference must be wrong. We choose the “smallest” and get a notation for a cut-free infinitary derivation which endsequent contains only wrong formulas and has an smaller ordinal. By iteration we get an infinite descending primitive recursive function since we may bound our search area in a primitive recursive way. Therefore $\exists y \in L_\omega \phi(s_n, y)$ must be true and with the help of a partial truth predicate we may transfer the result to the translation.

Let $\mathsf{dp}(m,n) := \mathsf{mod}(\mathsf{div}_2^{(m)}(n), 2)$ with $\mathsf{div}_2^{(0)}(n) := n$ and $\mathsf{div}_2^{(m+1)}(n) := \mathsf{div}(\mathsf{div}_2^{(m)}(n), 2)$ where $\mathsf{div}, \mathsf{mod}$ denote the usual number theoretic functions with $m = \mathsf{div}(m,k) \cdot k + \mathsf{mod}(m,k)$. For $a_0, \ldots, a_m \leq 1$ we have

$$\mathsf{dp}(i, \sum_{j=0}^{m} a_j \cdot 2^j) = 1 \quad \Leftrightarrow \quad i \leq m \text{ and } a_i = 1.$$

Definition 14.1. Definition of $\mathsf{Name}([x \in L_n : \psi(a_0, \ldots, a_m, x)])$ by induction on n and side induction on the rank of ψ:

$\mathsf{Name}(L_n) := 2_{n+1} - 1$
$\mathsf{Name}([x \in L_n : \mathrm{Ad}^\xi(x)]) := 0$
$\mathsf{Name}([x \in L_n : x \in x]) := 0$
$\mathsf{Name}([x \in L_n : x \in a]) := \mathsf{Name}(a)$
$\mathsf{Name}([x \in L_n : a \in x]) := \sum_{m=0}^{2_n - 1} \mathsf{dp}(\mathsf{Name}(a), m) \cdot 2^m$
$\mathsf{Name}([x \in L_n : \phi \wedge \psi]) := \mathsf{Name}([x \in L_n : \phi]) \cdot \mathsf{Name}([x \in L_n : \psi])$
$\mathsf{Name}([x \in L_n : \forall y \in L_n \psi(y)]) := \sum_{i=0}^{2_n - 1} (\prod_{j=0}^{2_n - 1} \mathsf{dp}(i, \mathsf{Name}([x \in L_n : \psi(s_j)]))) \cdot 2^i$
$\mathsf{Name}([x \in L_n : \neg\psi]) := \mathsf{Name}(L_n) - \mathsf{Name}([x \in L_n : \psi])$

where $\sum_{i=0}^{2_0 - 1}(\ldots) := 0$ and $\prod_{i=0}^{2_0 - 1}(\ldots) := 1$.

We may now define a partial truth predicate True_0 for $\Delta_0(\omega)$-sentences by means of Name:

Definition 14.2.

$\mathsf{true}_0(\mathrm{Ad}^\xi(t)) := 0$
$\mathsf{true}_0(s \in t) := \mathsf{dp}(\mathsf{Name}(s), \mathsf{Name}(t))$
$\mathsf{true}_0(\phi \wedge \psi) := \mathsf{true}_0(\phi) \cdot \mathsf{true}_0(\psi)$
$\mathsf{true}_0(\forall y \in t\psi(y)) := \prod_{j=0}^{2_{\mathsf{lev}(t)} - 1} \mathsf{sg}(1 - \mathsf{dp}(j, \mathsf{Name}(t)) + \mathsf{true}_0(\psi(s_j)))$
$\mathsf{true}_0(\neg\psi) := 1 - \mathsf{true}_0(\psi)$

where again $\prod_{i=0}^{2_0 - 1}(\ldots) := 1$.

Proposition 14.3. *We have*

1. $\mathsf{Name}(t) < 2_n \quad \Leftrightarrow \quad t \in \mathcal{T}_n \quad \Leftrightarrow \quad \mathsf{true}_0(t \in L_n) = 1$

2. $\mathsf{true}_0(t \in s) = \mathsf{true}_0(s \in L_m) = 1 \Rightarrow \mathsf{true}_0(t \in L_m) = 1$
3. *For* $m \geq \mathsf{lev}(t)$ *we have* $\mathsf{Name}(t) = \sum_{i=0}^{2_m - 1} \mathsf{true}_0(s_i \in t) \cdot 2^i$.
4. $\mathsf{true}_0(t \subseteq s) = 1 \quad \Rightarrow \quad \mathsf{Name}(t) \leq \mathsf{Name}(s)$
5. $\mathsf{true}_0(t = s) = 1 \quad \Rightarrow \quad \mathsf{Name}(t) = \mathsf{Name}(s)$
6. $\mathsf{Name}(t) = \mathsf{Name}(s) \quad \Rightarrow \quad \mathsf{true}_0(\phi(t)) = \mathsf{true}_0(\phi(s))$
7. $\mathsf{true}_0(t = s) = 1 \quad \Leftrightarrow \quad \mathsf{Name}(t) = \mathsf{Name}(s)$
8. $\mathsf{true}_0(s \in L_m) = \mathsf{true}_0(\phi(s)) = 1 \quad \Leftrightarrow \quad \mathsf{true}_0(s \in [x \in L_m : \phi(x)]) = 1$

Proof. 3. Natural induction on $\mathsf{lev}(t)$ with side induction on the skeleton of t. We have $\mathsf{true}_0(s_i \in t) = \mathsf{dp}(i, \mathsf{Name}(t))$. We consider only the cases $t = [x \in L_n : x \in a]$ and $t = [x \in L_n : a \in x]$:

$$\mathsf{Name}([x \in L_n : x \in a]) = \mathsf{Name}(a) \overset{I.H.}{=} \sum_{i=0}^{2_m - 1} \mathsf{dp}(i, \mathsf{Name}(a)) \cdot 2^i$$

$$\begin{aligned} \mathsf{Name}([x \in L_n : a \in x]) &= \sum_{j=0}^{2_n - 1} \mathsf{dp}(\mathsf{Name}(a), j) \cdot 2^i \\ &= \sum_{i=0}^{2_m - 1} \mathsf{dp}(i, \sum_{j=0}^{2_n - 1} \mathsf{dp}(\mathsf{Name}(a), j) \cdot 2^i) \cdot 2^i \end{aligned}$$

4. Let $m = \mathsf{lev}(t)$. Then follows

$$\begin{aligned} 1 &= \mathsf{true}_0(t \subseteq s) \\ &= \mathsf{true}_0(\forall x \in t . x \in s) \\ &= \prod_{i=0}^{2_m - 1} \mathsf{sg}(1 - \mathsf{dp}(i, \mathsf{Name}(t)) + \mathsf{true}_0(s_i \in s)) \\ &= \prod_{i=0}^{2_m - 1} \mathsf{sg}(1 - \mathsf{true}_0(s_i \in t) + \mathsf{true}_0(s_i \in s)) \end{aligned}$$

i.e.

$$\mathsf{true}_0(s_i \in t) = 1 \Rightarrow \mathsf{true}_0(s_i \in s) = 1.$$

By 3. follows the claim.
5. Follows from 4.
6. Induction on ϕ.
7. By main induction on $\mathsf{lev}(t)$ and side induction on the skeleton of t follows $\mathsf{true}_0(t = t) = 1$ and by 6. follows the claim.
8. Main induction on m, side induction on ϕ. ⊣

Proposition 14.4.

1. *For* $A \simeq \bigwedge(A_s)_{s \in \mathcal{T}_m}$ *we have*

$$\mathsf{true}_0(A) = 1 \quad \Leftrightarrow \quad \mathsf{true}_0(A_s) = 1 \text{ for all } s \in \mathcal{T}_m^*.$$

2. *For* $A \simeq \bigvee(A_s)_{s \in \mathcal{T}_m}$ *we have*

$$\mathsf{true}_0(A) = 1 \quad \Leftrightarrow \quad \mathsf{true}_0(A_s) = 1 \text{ for an } s \in \mathcal{T}_m^*.$$

PROOF. 1. Let $A \equiv t \notin [x \in L_m : \phi(x)]$. Then we have

$$
\begin{aligned}
\mathsf{true}_0(A) = 1 \quad &\Leftrightarrow \quad \mathsf{true}_0(\neg A) = 0 \\
&\overset{P.14.3.8.}{\Leftrightarrow} \mathsf{true}_0(t \in L_m) = 0 \text{ or } \mathsf{true}_0(\phi(t)) = 0 \\
&\overset{!}{\Leftrightarrow} \quad \mathit{for\,all}\ s \in \mathcal{T}_m^*\ \mathsf{true}_0(t = s) = 1 \Rightarrow \mathsf{true}_0(\phi(s)) = 0 \\
&\Leftrightarrow \quad \mathit{for\,all}\ s \in \mathcal{T}_m^*\ \mathsf{true}_0(t \neq s \vee \neg\phi(s)) = 1
\end{aligned}
$$

about (!): "$\Leftarrow$" Let $\mathsf{true}_0(t \in L_m) = 1$. Then we have $t \in \mathcal{T}_m$ and $s_{\mathsf{Name}(t)} \in \mathcal{T}_m^*$ as well as $\mathsf{true}_0(t = s_{\mathsf{Name}(t)}) = 1$ since $\mathsf{Name}(s_{\mathsf{Name}(t)}) = \mathsf{Name}(t)$. Therefore $0 = \mathsf{true}_0(\phi(s_{\mathsf{Name}(t)})) = \mathsf{true}_0(\phi(t))$.
"$\Rightarrow$" If $\mathsf{true}_0(t = s) = 1$ then follows $\mathsf{true}_0(\phi(s)) = \mathsf{true}_0(\phi(t)) = 0$.
For $A \equiv \forall x \in L_m \phi(x)$ the claim follows immediately by the definition of true_0.
2. Immediate from 1. ⊣

$$\mathsf{True}_0 := \{A \in \Delta_0(\omega) : \mathsf{true}_0(A) = 1\}, \mathsf{False}_0 := \Delta_0(\omega) \setminus \mathsf{True}_0.$$

DEFINITION 14.5. For $B = \exists x \in L_\omega A(x)$ with $A(L_0) \in \Delta_0(\omega)$ let

$$\mathbf{H}_\delta^*(B) := \{h \in \mathbf{H}_\delta : \mathsf{deg}(h) = 0 \text{ and } \mathsf{End}(h) \subseteq \{B\} \cup \mathsf{False}_0\}.$$

DEFINITION 14.6. Definition of $red(h)$ for $h \in \mathbf{H}_\delta^*(B)$

$$
red(h) := \begin{cases}
h & \text{if } \mathsf{tp}(h) = \bigwedge, A \simeq \bigwedge(A_s)_{s \in \mathcal{T}_0} \\
h[s_n] & \text{if } \mathsf{tp}(h) = \bigwedge_A, A \simeq \bigwedge(A_s)_{s \in \mathcal{T}_{m+1}}, \\
 & \quad n := \mu i < 2_{m+1}(\mathsf{true}_0(A_{s_i}) = 0) \\
h[n] & \text{if } \mathsf{tp}(h) = \bigwedge_{A_0 \wedge A_1}, n := \mu i < 2(\mathsf{true}_0(A_i) = 0) \\
h[0] & \text{if } \mathsf{tp}(h) = \bigvee_A^i
\end{cases}
$$

PROPOSITION 14.7. *If $A(s_n) \in \mathsf{False}_0$ for every natural number n then*

$$h \in \mathbf{H}_\delta^*(B) \Rightarrow red(h) \in \mathbf{H}_\delta^*(B) \text{ and } \mathsf{o}(red(h)) < \mathsf{o}(h).$$

PROOF. Let $\mathsf{tp}(h) = \bigvee_B^s$. Then is

$$\mathsf{End}(red(h)) = \mathsf{End}(h[0]) \subseteq \mathsf{End}(h) \cup \{A(s)\} \subseteq \{B\} \cup \mathsf{False}_0$$

by assumption and $\mathsf{true}_0(A(s)) = \mathsf{true}_0(A(s_m))$ for $m = \mathsf{Name}(s)$.
All other cases follow in the same way. ⊣

PROPOSITION 14.8. *If $A(s_n) \in \mathsf{False}_0$ for every natural number n then*

$$h \in \mathbf{H}_\delta^*(B) \Rightarrow red(h)^{(m)} \in \mathbf{H}_\delta^*(B)$$

for every natural number m.

PROOF. By natural induction on m from the preceding proposition. ⊣

PROPOSITION 14.9. *If $h \in \mathbf{H}_\delta^*(B)$ and there is an n with $\mathsf{o}(red^{n+1}(h)) \not< \mathsf{o}(red^n(h))$ then there is an n with $A(s_n) \in \mathsf{True}_0$.*

PROOF. By the two preceding propositions. ⊣

By means of Name we can translate $\Delta_0(\omega)$-formulas into arithmetical formulas. By $\bar{n}$ we denote the arithmetic term $\underbrace{S \ldots S}_{n-times} 0$.

Definition 14.10.

$$\begin{aligned}
&\mathcal{U}(x \in y) :\equiv \mathsf{dp}(x, \overline{y}) = 1\\
&\mathcal{U}(x \in a) :\equiv \mathsf{dp}(x, \overline{\mathsf{Name}(a)}) = 1\\
&\mathcal{U}(a \in b) :\equiv \mathsf{dp}(\overline{\mathsf{Name}(a)}, \overline{\mathsf{Name}(b)}) = 1\\
&\mathcal{U}(A \wedge B) :\equiv \mathcal{U}(A) \wedge \mathcal{U}(B)\\
&\mathcal{U}(\forall x \in yA) :\equiv \forall x < 2_{\mathsf{lev}(\mathsf{num}(y))}(\mathsf{dp}(x, y) = 0 \vee \mathcal{U}(A))\\
&\mathcal{U}(\forall x \in tA) :\equiv \forall x < 2_{\overline{\mathsf{lev}(t)}}(\mathsf{dp}(x, \overline{\mathsf{Name}(t)}) = 0 \vee \mathcal{U}(A))\\
&\mathcal{U}(\neg A) := \neg\mathcal{U}(A)
\end{aligned}$$

Note that we get an translation of set theoretical Δ_0-formulas into arithmetical Δ_0-formulas by this. Note further that the translation has the same free variables as the source formula. For the definition of num see the following pages. We are going to show that **PRA** proves that true_0 is a partial truth predicate for $\Delta_0(\omega)$-sentences. To emphasise all subtleties of the following arguments we make a precise difference at this point between all syntactic objects: formulas, terms, variables and Gödel numberings. That means we understand the functions Name and true_0 as defined on the Gödel numbers whereas the domain of $\mathcal{U}$ is still the set of $\Delta_0(\omega)$-formulas.
Since we need for the proof some elementary properties of the Gödel numbering of RS-terms and -sentences we define one specific Gödel numbering below. We assume we have already a Gödel numbering of the logical and non logical symbols as well of the variables and $\in$-formulas.

Definition 14.11.

$$\begin{aligned}
&\mathsf{gn}(s \in t) := \langle \mathsf{gn}(\in), \mathsf{gn}(s), \mathsf{gn}(t)\rangle\\
&\mathsf{gn}(s \notin t) := \langle \mathsf{gn}(\notin), \mathsf{gn}(s), \mathsf{gn}(t)\rangle\\
&\mathsf{gn}(A \wedge B) := \langle \mathsf{gn}(\wedge), \mathsf{gn}(A), \mathsf{gn}(B)\rangle\\
&\mathsf{gn}(A \vee B) := \langle \mathsf{gn}(\vee), \mathsf{gn}(A), \mathsf{gn}(B)\rangle\\
&\mathsf{gn}(\forall x \in tA) := \langle \mathsf{gn}(\forall), \mathsf{gn}(x), \mathsf{gn}(t), \mathsf{gn}(A)\rangle\\
&\mathsf{gn}(\exists x \in tA) := \langle \mathsf{gn}(\exists), \mathsf{gn}(x), \mathsf{gn}(t), \mathsf{gn}(A)\rangle\\
&\mathsf{gn}([x \in L_n : \phi(a_1, \ldots, a_m, x)]) := \langle \mathsf{gn}(L_n), \mathsf{gn}(x), \mathsf{gn}(\phi), \mathsf{gn}(a_1), \ldots, \mathsf{gn}(a_m)\rangle
\end{aligned}$$

The following proposition shall remind the reader of some elementary facts:

Proposition 14.12.

1. **PRA** $\vdash x \leq \bar{n} \leftrightarrow x = \bar{0} \vee \ldots \vee x = \bar{n}$
2. **PRA** $\vdash \prod_{x=\bar{0}}^{\bar{n}} f(x) = 1 \leftrightarrow f(\bar{0}) = 1 \wedge \ldots \wedge f(\bar{n}) = 1$
3. **PRA** $\vdash \sum_{x=\bar{0}}^{\bar{n}} f(x) \geq 0 \leftrightarrow f(\bar{0}) \geq 1 \vee \ldots \vee f(\bar{n}) \geq 1$
4. **PRA** $\vdash \forall x \leq \bar{n}\phi(x) \leftrightarrow \phi(\bar{0}) \wedge \ldots \wedge \phi(\bar{n})$
5. **PRA** $\vdash \exists x \leq \bar{n}\phi(x) \leftrightarrow \phi(\bar{0}) \vee \ldots \vee \phi(\bar{n})$

□

Let num a[2] primitive recursive function with $\mathsf{num}(n) := \mathsf{gn}(s_n)$ and

$$\mathbf{PRA} \vdash \mathsf{Name}(\mathsf{num}(x)) = x.$$

Let further sb be a primitive recursive function with

$$\mathsf{sb}(\mathsf{gn}(A), \mathsf{gn}(x), \mathsf{gn}(t)) = \mathsf{gn}(A_x(t))$$

for RS-Formulas A, variables x and RS-terms t. We write $A(\dot{x})$ for the arithmetical term

$$\mathsf{sb}(\overline{\mathsf{gn}(A)}, \overline{\mathsf{gn}(x)}, \mathsf{num}(x)).$$

Note that the variable x in this term occurs free just at the last place [3].
With sb we may define true_0 on the Gödel numbers of $\Delta_0(\omega)$-sentences as follows:

DEFINITION 14.13.

$\mathsf{true}_0(\langle \mathsf{gn}(\in), x, y\rangle) := \mathsf{dp}(\mathsf{Name}(x), \mathsf{Name}(y))$
$\mathsf{true}_0(\langle \mathsf{gn}(\wedge), x, y\rangle) := \mathsf{true}_0(x) \cdot \mathsf{true}_0(y)$
$\mathsf{true}_0(\langle \mathsf{gn}(\forall), x, y, z\rangle) := \prod_{j=0}^{2_{\mathsf{lev} y}-1} \mathsf{sg}((1 - \mathsf{dp}(j, \mathsf{Name}(y)) + \mathsf{true}_0(\mathsf{sb}(z, x, \mathsf{num}(j))))$

The reason for this detailed presentation which may look exaggerated is that we need some rather profound properties of the functions $\mathsf{true}_0, \mathsf{gn}$ and sb in the proof of the next proposition. The next proposition states that we can say something about the function values $\mathsf{true}_0(x)$ in **PRA** without completely knowing the argument x. It is easy to see that our definitions of the functions $\mathsf{true}_0, \mathsf{gn}$ and sb have the properties needed in the proof of the following proposition.

PROPOSITION 14.14. *For $\Delta_0(\omega)$-formulas A we have*

$$\mathbf{PRA} \vdash \mathsf{true}_0(A(\dot{x}_0, \ldots, \dot{x}_m)) = 1 \leftrightarrow \mathcal{U}(A(x_0, \ldots, x_m))$$

PROOF. Let $A \equiv \phi(a_0, \ldots, a_n, x_0, \ldots, x_m)$. Structural induction on ϕ.
We just look at the cases with the for all quantifiers. Further we omit the RS-terms $a_0, \ldots, a_n$ since they play no particular rôle in the argumentation.
Let $\phi(x_0, \ldots, x_m) \equiv \forall x_{m+1} \in t.\psi(x_0, \ldots, x_{m+1})$.
By I.H. we have

$$\mathbf{PRA} \vdash \mathsf{true}_0(\psi(\dot{x}_0, \ldots, \dot{x}_{m+1})) = 1 \leftrightarrow \mathcal{U}(\psi(x_0, \ldots, x_{m+1})).$$

[2]For the proof we need some properties which depend on the intensional way the function is given. This is the reason why we write “a” instead of “the” here and at some other places. From an extensional point of view there is only one function.

[3]Such formulations which I think are rather ill-chosen are common in the literature see e.g. [11, 29]. However they allow to emphasise the double rôle of x (as metavariable and variable of the term) by the formulation $A(\dot{x})$.

Therefore we may argue in **PRA** as follows:

$$\begin{aligned} 1 &= \mathsf{true}_0(\phi(\dot{x}_0, \ldots, \dot{x}_m)) \\ &= \mathsf{true}_0(\langle \overline{\mathsf{gn}(\forall)}, \overline{\mathsf{gn}(x_{m+1})}, \overline{\mathsf{gn}(t)}, \psi(\dot{x}_0, \ldots, \dot{x}_m)\rangle) \\ &= \prod_{x_{m+1}=0}^{2_{\mathsf{lev}(\overline{\mathsf{gn}(t)})}-1} \mathsf{sg}(1 - [x_{m+1} \in t] + \mathsf{true}_0(\psi(\dot{x}_0, \ldots, \dot{x}_{m+1}))) \end{aligned}$$

$$\Leftrightarrow$$

$$\forall x_{m+1} < 2_{\mathsf{lev}(\overline{\mathsf{gn}(t)})}[x_{m+1} \in t] = 1 \rightarrow \mathsf{true}_0(\psi(\dot{x}_0, \ldots, \dot{x}_{m+1}))$$

$$\overset{I.H.}{\Leftrightarrow}$$

$$\forall x_{m+1} < 2_{\overline{\mathsf{lev}(t)}}[x_{m+1} \in t] = 1 \rightarrow \mathcal{U}(\psi(x_0, \ldots, x_{m+1}))$$

$$\Leftrightarrow$$

$$\mathcal{U}(\phi(x_0, \ldots, x_m))$$

where $[x_{m+1} \in t] := \mathsf{dp}(x_{m+1}, \mathsf{Name}(\overline{\mathsf{gn}(t)}))$.
In the case $\phi(x_0, \ldots, x_m) \equiv \forall x_{m+1} \in x_j \psi(x_0, \ldots, x_{m+1})$ we get:

$$\begin{aligned} 1 &= \mathsf{true}_0(\phi(\dot{x}_0, \ldots, \dot{x}_m)) \\ &= \mathsf{true}_0(\langle \overline{\mathsf{gn}(\forall)}, \overline{\mathsf{gn}(x_{m+1})}, \mathsf{num}(x_j), \psi(\dot{x}_0, \ldots, \dot{x}_m)\rangle) \\ &= \prod_{x_{m+1}=0}^{2_{\mathsf{lev}(\mathsf{num}(x_j))}-1} \mathsf{sg}(1 - [x_{m+1} \in x_j] + \mathsf{true}_0(\psi(\dot{x}_0, \ldots, \dot{x}_{m+1}))) \end{aligned}$$

$$\Leftrightarrow$$

$$\forall x_{m+1} < 2_{\mathsf{lev}(\mathsf{num}(x_j))}[x_{m+1} \in x_j] = 1 \rightarrow \mathsf{true}_0(\psi(\dot{x}_0, \ldots, \dot{x}_{m+1}))$$

$$\overset{I.H.}{\Leftrightarrow}$$

$$\forall x_{m+1} < 2_{\mathsf{lev}(\mathsf{num}(x_j))}[x_{m+1} \in x_j] = 1 \rightarrow \mathcal{U}(\psi(x_0, \ldots, x_{m+1}))$$

$$\Leftrightarrow$$

$$\mathcal{U}(\phi(x_0, \ldots, x_m))$$

since $\mathbf{PRA} \vdash \mathsf{Name}(\mathsf{num}(x_j)) = x_j$
where $[x_{m+1} \in x_j] := \mathsf{dp}(x_{m+1}, \mathsf{Name}(\mathsf{num}(x_j)))$. ⊣

Corollary. For $\Delta_0(\omega)$-sentences A we have

$$\mathbf{PRA} \vdash \mathsf{true}_0(\overline{\mathsf{gn}(A)}) = 1 \leftrightarrow \mathcal{U}(A).$$

The functions $\mathsf{o}, \mathsf{deg}, \mathsf{ref}, \mathsf{tp}, []$ and the predicate $\mathbf{H}_\delta$ are defined in a primitive recursive way (where the recursion is on the length of the argument: the string d). The proof of the arithmetical proposition that $\mathbf{H}_\delta$ is a notation system for $RS(\mathcal{K})$-derivations i.e. the verification of the Δ_0-properties a)-n) just uses Δ_0-induction as do the proofs of all propositions needed in the proof. For a proof that the functions true_0 and Name are primitive recursive see [17]. Therefore we are able to prove our last theorem:

Theorem 14.15. *From* $\mathbf{KP} + \Pi_3\text{-Refl} \vdash \forall z(\text{“z=HF”} \rightarrow \forall x \in z \exists y \in z \phi(x, y))$ *with* $\phi \in \Delta_0$ *follows* $\mathbf{PRA} + \mathrm{PRWO}(<) \vdash \forall n \exists m \mathcal{U}(\phi(n, m))$.

PROOF. We denote the function symbol representing the primitive recursive function I with $\mathsf{I}(\mathsf{gn}(A), \mathsf{gn}(t), \mathsf{gn}(h)) = \mathsf{gn}(\mathrm{I}_t^A h)$ by I as well.
Let $\mathbf{KP} + \Pi_3$-Refl $\vdash \forall z(\text{“z=HF”} \rightarrow \forall x \in z \exists y \in z \phi(x, y))$ where $\phi \in \Delta_0$.
By Theorem 12.5 there is a δ and a $h \in \mathbf{H}_\delta$ with $\mathsf{deg}(h) = 0$ and

$$\mathsf{End}(h) \subseteq \{\forall x \in L_\omega \exists y \in L_\omega \phi(x, y)\}.$$

Let $h(\dot{n})$ denote $\mathsf{I}(\overline{\mathsf{gn}(\forall x \in L_\omega \exists y \in L_\omega \phi(x, y)}, \mathsf{num}(n), \overline{\mathsf{gn}(h)})$.
We have

$$\mathbf{PRA} \vdash h(\dot{n}) \in \mathbf{H}^*_\delta(\exists y \in L_\omega \phi(\dot{n}, y))$$

and

$$\mathbf{PRA} + \mathrm{PRWO}(<) \vdash \exists m \ \mathsf{o}(red^{m+1}(h(\dot{n}))) \not< \mathsf{o}(red^m(h)).$$

By proposition 14.9 follows

$$\mathbf{PRA} + \mathrm{PRWO}(<) \vdash \exists m \ \mathsf{true}_0(\phi(\dot{n}, \dot{m})) = 1$$

and since $\mathbf{PRA} \vdash \mathsf{true}_0(\phi(\dot{n}, \dot{m})) = 1 \leftrightarrow \mathcal{U}(\phi(n, m))$ therefore

$$\mathbf{PRA} + \mathrm{PRWO}(<) \vdash \exists m \mathcal{U}(\phi(n, m)).$$

By generalisation follows the claim. ⊣

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Appendix A. Notations for the sub derivations. Remember that the inferences are only used in the situations of Definition 12.1. We write $d : \Gamma$ for $\mathsf{End}(d) \subseteq \Gamma$.

Ad $h := (\mathrm{N1})^{\xi,\pi}_{B[\vec{s}]}(t)$:

$\Gamma(\vec{s}, L_\tau) := \neg\psi^\pi_1, \dots, \neg\psi^\pi_l, \neg(\mathrm{tran}(L_\tau) \wedge L_\tau \neq \emptyset \wedge Pair^\tau), \neg C[\vec{s}], B[\vec{s}]$

$d_{b_2} :=$

$$\cfrac{1 : \psi^\pi_l \qquad \cfrac{\cfrac{1 : \psi^\pi_1 \quad d_1(\vec{a}/\vec{s}, y/L_\tau) : \Gamma(\vec{a}/\vec{s}, y/L_\tau)}{} \ \mathrm{Cut}_{\psi^\pi_1}}{\vdots}}{\Gamma(\vec{a}/\vec{s}, y/L_\tau) \setminus \{\neg\psi^\pi_1, \dots, \neg\psi^\pi_l\}} \ \mathrm{Cut}_{\psi^\pi_l}$$

$d_b :=$

$$\dfrac{\dfrac{d_{b_2} : \Gamma(\vec{a}/\vec{s}, y/L_\tau) \setminus \{\neg\psi_1^\pi, \dots, \neg\psi_l^\pi\} \quad d_{b_1} : \mathrm{tran}(L_\tau) \wedge L_\tau \neq \emptyset \wedge Pair^\tau}{\neg C[\vec{s}]^\tau, B[\vec{s}]^\tau}\ \mathrm{Cut} \quad \dfrac{3 : L_\tau = L_\tau}{\mathrm{Ad}^\xi(L_\tau)}}{\dfrac{\vdots}{\neg C[\vec{s}]^\tau, \exists z^\pi(\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z)}}$$

$h[\tau] :=$

$$\dfrac{d_b : \neg C[\vec{s}]^\tau, \exists z^\pi(\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z) \quad 10 : L_\tau \neq t, \neg C[\vec{s}]^t, C[\vec{s}]^\tau}{L_\tau \neq t, \neg C[\vec{s}]^t, \exists z^\pi(\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z)}\ \mathrm{Cut}$$

By $\bigwedge_{\neg\mathrm{Ad}^\xi(t)} = \mathsf{tp}(h)$ follows the conclusion of h

$$\neg\mathrm{Ad}^\xi(t), \neg C[\vec{s}]^t, \exists z^\pi(\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z).$$

Ad $h := (\mathrm{N}2)^{\xi,\pi}_{B[\vec{s}]}$:

$h[t] :=$

$$\dfrac{\dfrac{\neg\mathrm{Ad}^\xi(t), \neg C[\vec{s}]^t, \exists z^\pi(\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z)}{\vdots}}{\neg\mathrm{Ad}^\xi(t) \wedge \neg C[\vec{s}]^t, \exists z^\pi(\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z)}$$

By $\mathsf{tp}(h) = \bigwedge_{\forall z^\pi(\neg\mathrm{Ad}^\xi(z) \vee \neg C[\vec{s}]^z)}$ follows

$$\forall z^\pi(\neg\mathrm{Ad}^\xi(z) \vee \neg C[\vec{s}]^z), \exists z^\pi(\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z).$$

Let $\Gamma := \mathsf{End}(h_0)$.

Ad $h := ((8.9)^{\xi,\pi}_A)h_0$:

$\mathsf{tp}(h_0) = \bigvee^s_A$:

$h[0] :=$

$$\dfrac{h_0[0] : A_s, \Gamma}{\exists z^\pi(\mathrm{Ad}^\xi(z) \wedge A^{(z,\pi)}), (A_s, \Gamma) \setminus \{A\}}\ (8.9)^{\xi,\pi}_A$$

By $\mathsf{tp}(h) = \mathrm{Ref}^\xi_\pi A(s)$ follows

$$\exists z^\pi(\mathrm{Ad}^\xi(z) \wedge A^{(z,\pi)}), \Gamma \setminus \{A\}.$$

Ad $h := ((8.10)^{\xi,\pi}_{A_1,\dots,A_n})h_0$:

Note that $A \equiv A_1 \wedge \ldots \wedge A_n \equiv B[\vec{s}]^\pi$ (see p. 24).

$d :=$

$$\dfrac{1 : \phi_k^\pi \qquad \dfrac{1 : \phi_1^\pi \quad d_0(\vec{a}/\vec{s}) : \neg\phi_1^\pi, \dots \neg\phi_k^\pi, \neg B[\vec{s}]^\pi, C[\vec{s}]^\pi}{\vdots}\ \mathrm{Cut}_{\phi_1^\pi}}{\neg B[\vec{s}]^\pi, C[\vec{s}]^\pi}\ \mathrm{Cut}_{\phi_k^\pi}$$

$h[0] :=$

$$\dfrac{h_0 : \Gamma \quad \dfrac{d : \neg B[\vec{s}]^\pi, C[\vec{s}]^\pi}{\neg B[\vec{s}]^\pi, \exists z^\pi(\mathrm{Ad}^\xi(z) \wedge C[\vec{s}]^z)}\ (8.9)^{\xi,\pi}_{C[\vec{s}]^\pi}}{\exists z^\pi(\mathrm{Ad}^\xi(z) \wedge C[\vec{s}]^z), \Gamma \setminus \{B[\vec{s}]\})}\ \mathrm{Cut}_{B[\vec{s}]}$$

$h[1] :=$

$$(\mathrm{N2})^{\xi,\pi}_{B[\vec{s}]} : \neg(\exists z^\pi(\mathrm{Ad}^\xi(z) \wedge C[\vec{s}]^z)), \exists z^\pi(\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z)$$

By $\mathsf{tp}(h) = \mathrm{Cut}_{\exists z^\pi(\mathrm{Ad}^\xi(z) \wedge C[\vec{s}]^z)}$ follows

$$\exists z^\pi(\mathrm{Ad}^\xi(z) \wedge B[\vec{s}]^z), \Gamma \setminus \{B[\vec{s}]\}.$$

Ad $h := ((\mathrm{H}_1 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B})h_0$:

Note that $C \equiv \exists u^{\mathcal{K}}(\mathrm{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})})$.

$h[s] :=$

$$\dfrac{\dfrac{h_0 : \Gamma, B}{\neg \mathrm{Ad}^\alpha(s), \bigvee \Gamma^{(s,\mathcal{K})}, C^{(\pi,\mathcal{K})}}\ (\mathrm{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s)}{\dfrac{\vdots}{\neg \mathrm{Ad}^\alpha(s) \vee \bigvee \Gamma^{(s,\mathcal{K})}, C^{(\pi,\mathcal{K})}}}$$

By $\mathsf{tp}(h) = \bigwedge_{\forall v^\pi \neg \mathrm{Ad}^\alpha(v) \vee \bigvee \Gamma^{(v,\mathcal{K})}}$ follows

$$\forall v^\pi(\neg \mathrm{Ad}^\alpha(v) \vee \bigvee \Gamma^{(v,\mathcal{K})}), C^{(\pi,\mathcal{K})}.$$

Ad $h := ((\mathrm{H}_2 10.1)^{\pi,\alpha}_{\gamma,\Gamma,B}(s))h_0$:

Note that $C \equiv \exists u^{\mathcal{K}}(\mathrm{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})})$
and $G \equiv \mathrm{tran}(L_\tau) \wedge L_\tau \neq \emptyset \wedge B^{(\tau,\mathcal{K})}$.

$$\dfrac{d_1 : L_\tau \neq s, \bigvee \Gamma^{(s,\mathcal{K})}, \bigwedge \neg \Gamma^{(\tau,\mathcal{K})} \quad \dfrac{\dfrac{d_2 : \mathrm{tran}(L_\tau) \wedge L_\tau \neq \emptyset \quad \dfrac{\vdots \quad \dfrac{h_0 : \Gamma, B}{\Gamma^{(\tau,\mathcal{K})}, B^{(\tau,\mathcal{K})}}\ (10.1)^\tau_{\gamma,\Gamma,B}}{\bigvee \Gamma^{(\tau,\mathcal{K})}, B^{(\tau,\mathcal{K})}}}{\bigvee \Gamma^{(\tau,\mathcal{K})}, G}}{\bigvee \Gamma^{(\tau,\mathcal{K})}, C^{(\pi,\mathcal{K})}}}{L_\tau \neq s, \bigvee \Gamma^{(s,\mathcal{K})}, C^{(\pi,\mathcal{K})}}\ \mathrm{Cut}$$

By $\mathsf{tp}(h) = \bigwedge_{\neg \mathrm{Ad}^\alpha(s)}$ follows

$$\neg \mathrm{Ad}^\alpha(s), \bigvee \Gamma^{(s,\mathcal{K})}, C^{(\pi,\mathcal{K})}.$$

Ad $h := ((10.1)^{\pi}_{\gamma,\Gamma})h_0$:

$\mathsf{tp}(h_0) = \mathrm{Ref}_{\mathcal{K}}(B)$:
Note that $C \equiv \exists u^{\mathcal{K}}(\mathrm{tran}(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})}) \in \Gamma = \mathsf{End}(h_0)$ because of $\mathsf{tp}(h_0) = \mathrm{Ref}_{\mathcal{K}}(B)$.

$h[0] :=$

$$\frac{d_0 : \bigwedge \neg\Gamma^{(\pi,\mathcal{K})}, \Gamma^{(\pi,\mathcal{K})}}{\exists z^{\pi}(\mathrm{Ad}^{\hat{\alpha}_0}(z) \wedge \bigwedge \neg\Gamma^{(z,\mathcal{K})}), \Gamma^{(\pi,\mathcal{K})}} \ (8.10)^{\pi,\hat{\alpha}_0}_{\neg\Gamma^{(\pi,\mathcal{K})}}$$

$h[1] :=$

$$\frac{h_0[0] : \Gamma, B}{\neg\exists z^{\pi}(\mathrm{Ad}^{\hat{\alpha}_0}(z) \wedge \bigwedge \neg\Gamma^{(z,\mathcal{K})}), C^{(\pi,\mathcal{K})}} \ (\mathrm{H}_1 10.1)^{\pi\hat{\alpha}_0}$$

By $\mathsf{tp}(h) = \mathrm{Cut}_{\exists z^{\pi}(\mathrm{Ad}^{\hat{\alpha}_0}(z) \wedge \bigwedge \neg G^{(z,\mathcal{K})})}$ follows

$$\Gamma^{(\pi,\mathcal{K})}$$

Ad $h := ((\mathrm{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta} A(s))h_0$:

For $A(s) \equiv \forall y^{\pi} \exists x^{\pi} G(s, y, x)$ we have $A(s)_t \equiv \exists x^{\pi} G(s, t, x)$.
Let $\Gamma := \mathsf{End}(h_0) \setminus \{A(s)\}$.

$h[t] :=$

$$\cfrac{\cfrac{\cfrac{h_0 : \Gamma, \forall y^{\pi} \exists x^{\pi} G(s, y, x)}{\Gamma, \exists x^{\pi} G(s, t, x)} \ \mathrm{I}^{A(s)}_{t}}{\Gamma, \exists x^{\pi} G(s, t, x)} \ (10.2)^{\mu,\pi,\sigma}_{\gamma_t}}{\Gamma, \exists x^{\eta} G(s, t, x)} \ \mathrm{B}^{(\eta,\pi)}_{A(s)_t}$$

By $\mathsf{tp}(h) = \bigwedge_{A(s)^{(\eta,\pi)}}$ follows $\Gamma, A(s)^{(\eta,\pi)}$.

Ad $h = ((10.2)^{\mu,\pi,\xi}_{\gamma})h_0$:

$\mathsf{tp}(h_0) = \mathrm{Ref}^{\sigma}_{\pi} A(s)$:
Let $\Gamma := \mathsf{End}(h_0) \setminus \{A(s)\}$, $A(s)$ as above.

$h[0] :=$

$$\cfrac{\cfrac{L_\eta = L_\eta}{\mathrm{Ad}^{\sigma}(L_\eta)} \qquad \cfrac{\cfrac{h_0[0] : \Gamma, \forall y^{\pi} \exists x^{\pi} G(s, y, x)}{\Gamma, \forall y^{\eta} \exists x^{\eta} G(s, y, x)} \ (\mathrm{H}10.2)^{\mu,\pi,\sigma}_{\gamma,\zeta}}{\Gamma, \exists u^{\eta} \forall y^{\eta} \exists x^{\eta} G(u, y, x)}}{\Gamma, \mathrm{Ad}^{\sigma}(L_\eta) \wedge \exists u^{\eta} \forall y^{\eta} \exists x^{\eta} G(u, y, x)}$$

By $\mathsf{tp}(h) = \bigvee^{\eta}_{\exists z^{\pi}(\mathrm{Ad}^{\sigma}(z) \wedge \exists u^{z} \forall y^{z} \exists x^{z} G(u,y,x))}$ follows $\mathsf{End}(h_0)$.

$\mathsf{tp}(h_0) = \mathrm{Cut}_A$:
Let $\Gamma_0 := \mathsf{End}(h_0[0]) \setminus \{A\}$ and $\Gamma_1 := \mathsf{End}(h_0[1]) \setminus \{\neg A\}$.

$rk(A) = \mathcal{K}$:

$h[0] :=$

$$\cfrac{\cfrac{h_0[0] : \Gamma_0, A}{\Gamma_0, A^{(\kappa,\mathcal{K})}}\ (10.1)^{\kappa}_{\gamma,\Gamma,A} \quad \cfrac{h_0[1] : \Gamma_1, \neg A}{\Gamma_1, \neg A^{(\kappa,\mathcal{K})}}\ (10.1)^{\kappa}_{\gamma,\Gamma,\neg A}}{\cfrac{\mathsf{End}(h_0)}{\mathsf{End}(h_0)}\ (10.2)^{\mu',\pi,\xi}_{\gamma'}}\ \mathrm{Cut}_{A^{(\kappa,\mathcal{K})}}$$

By Rep_0 follows $\mathsf{End}(h_0)$.

$\pi < rk(A) \notin Reg$:

$h[0] :=$

$$\cfrac{\cfrac{h_0[0] : \Gamma_0, A}{\Gamma_0, A}\ (10.2)^{\nu,\tau,0}_{\gamma} \quad \cfrac{h_0[1] : \Gamma_1, \neg A}{\Gamma_1, \neg A}\ (10.2)^{\nu,\tau,0}_{\gamma}}{\cfrac{\cfrac{\mathsf{End}(h_0)}{\mathsf{End}(h_0)}\ \mathrm{E}^{\Psi^0_\tau(\hat{\alpha}_0)}_{\bar{\nu}}}{\mathsf{End}(h_0)}\ (10.2)^{\nu,\pi,\xi}_{\hat{\alpha}_0}}\ \mathrm{Cut}_A$$

By Rep_0 follows $\mathsf{End}(h_0)$.

$\pi \leq rk(A) \in Reg$, $\alpha_0 < rk(A) =: \tau$, without lost of generality $A \equiv \exists x^\tau \neg F(x)$:

$h[0] :=$

$$\cfrac{\cfrac{h_0[1] : \mathsf{End}(h_0), \forall x^\tau F(x)}{\mathsf{End}(h_0)}\ S_{\neg A}}{\mathsf{End}(h_0)}\ (10.2)^{\mu,\pi,\xi}_{\gamma}$$

By Rep_0 follows $\mathsf{End}(h_0)$.

$\pi \leq rk(A) \in Reg$, $\pi = \tau \leq \alpha_0$, without lost of generality $A \equiv \exists x^\tau \neg F(x)$:

$h[0] :=$

$$\cfrac{\cfrac{\cfrac{h_0[0] : \Gamma_0, \exists x^\tau \neg F(x)}{\Gamma_0, \exists x^\tau \neg F(x)}\ (10.2)^{\mu,\tau,0}_{\gamma}}{\Gamma_0, \exists x^{\Psi^0_\tau(\hat{\alpha}_0)} \neg F(x)}\ \mathrm{B}^{(\Psi^0_\tau(\hat{\alpha}_0),\tau)}_{A} \quad \cfrac{\cfrac{h_0[1] : \Gamma_1, \forall x^\tau F(x)}{\Gamma_1, \forall x^{\Psi^0_\tau(\hat{\alpha}_0)} F(x)}\ \forall^{\Psi^0_\tau(\hat{\alpha}_0)}_{w} F(x)}{\Gamma_1, \forall x^{\Psi^0_\tau(\hat{\alpha}_0)} F(x)}\ (10.2)^{\mu,\tau,0}_{\hat{\alpha}_0}}{\mathsf{End}(h_0)}\ \mathrm{Cut}_{A^{(\Psi^0_\tau(\hat{\alpha}_0),\tau)}}$$

By Rep_0 follows $\mathsf{End}(h_0)$.

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