Wheels

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Abstract

The following is not very well checked yet, i.e. there might be some typoi and mistakes.

Definition 1 (**Quotientwheel of an integral domain**)
Let $\mathrm{R} := (|\mathrm{R}|, 0, 1, +, \cdot, -)$ be an integral domain (integral domain always with $0 \neq 1$). We write R for $|\mathrm{R}|$.

(a) Define $\sim$ on R^2 by $(a, b) \sim (a', b') \Leftrightarrow \exists r, r' \in \mathrm{R}.r \neq 0 \wedge r' \neq 0 \wedge (a \cdot r, b \cdot r) = (a' \cdot r', b' \cdot r')$.

(b) Let $\frac{a}{b}$ be the equivalence class of (a, b) modulo $\sim$.

(c) Let $\mathrm{R}^\infty_\perp := (|\mathrm{R}^\infty_\perp|, 0', 1', +', \cdot', -', (\frac{1}{\cdot})', \infty, \perp)$, the *quotient wheel of* R (name preliminary), be defined by:
$|\mathrm{R}^\infty_\perp| = \{\frac{a}{b} \mid a, b \in \mathrm{R}\}$.
$\frac{a}{b} +' \frac{a'}{b'} := \frac{a \cdot b' + a' \cdot b}{b \cdot b'}$.
$\frac{a}{b} \cdot' \frac{a'}{b'} := \frac{a \cdot b}{a' \cdot b'}$.
$-' \frac{a}{b} := \frac{-a}{b}$.
$(\frac{1}{\frac{a}{b}})' := \frac{b}{a}$.
$\infty := \frac{1}{0}$.
$\perp := \frac{0}{0}$.

*This article was insprired by disussions during Jens Blanck's lectures [Bla97] on [Pot97]. The idea to extend the quotient field by allowing fractions with denominator 0 is due to P. Martin-Löf.

(d) We write $R_\perp^\infty$ instead of $|R_\perp^\infty|$.
Let $\iota : R \to R_\perp^\infty$, $\iota(a) := \frac{a}{1}$. We identify a with $\iota(a)$, R with $\iota[R]$, and omit $'$ and sometimes $\cdot$. (Note $0' = \frac{0}{1} = \iota(0)$, $1' = \iota(1)$).

Lemma 2 *(a)* $\frac{a}{b} = \iota(a) \cdot (\frac{1}{\cdot})'(\iota(b))$.

(b) $\mathrm{R}_\perp^\infty = \mathrm{Quot}(\mathrm{R})$ *is the quotient field of* R*, then* $\mathrm{R}_\perp^\infty = \mathrm{Quot}(\mathrm{R}) \cup \{\infty, \perp\}$.

(c) Let $a, b \in \mathrm{Quot}(\mathrm{R})$*. Then the following "multiplication tables" hold in* $\mathrm{R}_\perp^\infty$*:*

$+$	a	∞	$\perp$
b	$a+b$	∞	$\perp$
∞	∞	$\perp$	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$

$\cdot$	$a \neq 0$	0	∞	$\perp$
$b \neq 0$	ab	0	∞	$\perp$
0	0	0	$\perp$	$\perp$
∞	∞	$\perp$	∞	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

	$\frac{1}{x}$	$-x$
$a \neq 0$	$\frac{1}{a}$	$-a$
0	∞	0
∞	0	∞
$\perp$	$\perp$	$\perp$

Proof:
(a) $a \cdot (\frac{1}{\cdot})'(b) = \frac{a}{1} \cdot \frac{1}{b} = \frac{a}{b}$.
(b) Only $\subset$: If $b \neq 0$, then $\frac{a}{b} = \frac{a \cdot \frac{1}{b}}{b \cdot \frac{1}{b}} = \iota(a \cdot \frac{1}{b})$.
$\frac{a}{0} = \frac{1}{0} = \infty$ if $a \neq 0$.
$\frac{0}{0} = \perp$.
(c): Verify.

Definition 3 (a) A *wheel* is an n-tupel

$$\mathrm{R}_\perp^\infty = (|\mathrm{R}_\perp^\infty|, 0, 1, +, \cdot, -, \frac{1}{\cdot}, \infty, \perp) \ ,$$

where $|\mathrm{R}_\perp^\infty|$ is a set (written as $\mathrm{R}_\perp^\infty$), 0, 1, ∞, $\perp \in \mathrm{R}_\perp^\infty$, $-, \frac{1}{\cdot}$ are unary functions on $\mathrm{R}_\perp^\infty$, and $+, \cdot$ are binary functions on $\mathrm{R}_\perp^\infty$, written in the usual way, such that for all $x, y, z \in \mathrm{R}_\perp^\infty$ the following axioms hold:

Commutativity	$x + y = y + x.$
	$x \cdot y = y \cdot x.$
Associativity	$(x + y) + z = x + (y + z).$
	$(x \cdot y) \cdot z = x \cdot (y \cdot z).$
Distributivity	$z \neq \infty \to (x + y) \cdot z = x \cdot z + y \cdot z,$
Neutral Elements	$x + 0 = x$
	$x \cdot 1 = x.$
Inverse Elements	$x \notin \{\infty, \perp\} \to x + (-x) = 0.$
	$x \notin \{0, \infty, \perp\} \to x \cdot \frac{1}{x} = 1.$
Definition of ∞, $\perp$	$\infty = \frac{1}{0}.$
	$\perp = 0 \cdot \infty$
Strictness	$x \cdot \perp = \perp$
	$x + \perp = \perp.$
	$- \perp = \perp.$
	$\frac{1}{\perp} = \perp.$
Laws for ∞	$x \notin \{\infty, \perp\} \to x + \infty = \infty.$
	$\infty + \infty = \perp.$
	$x \notin \{0, \perp\} \to x \cdot \infty = \infty.$
	$-\infty = \infty.$
	$\frac{1}{\infty} = 0.$
Non-triviality	$0 \neq 1.$

(b) $\frac{x}{y} = x \cdot \frac{1}{y}$,
$x - y := x + (-y)$.
The usual conventions like omitting of $\cdot$, omitting parentheses (where $\cdot$ binds more than $+$) apply.

Remark 4 *The following axioms*

Strictness	$x\cdot\bot=\bot$
	$x+\bot=\bot.$
	$-\bot=\bot.$
	$\frac{1}{\bot}=\bot.$
Laws for ∞	$x\notin\{\infty,\bot\}\to x+\infty=\infty.$
	$\infty+\infty=\bot.$
	$x\notin\{0,\bot\}\to x\cdot\infty=\infty.$
	$-\infty=\infty.$
	$\frac{1}{\infty}=0.$

can be replaced by the following axioms:

Laws of $\frac{1}{\cdot}$	$\frac{1}{\frac{1}{x}}=x$
	$\frac{1}{xy}=\frac{1}{x}\frac{1}{y}.$
	$y\neq\infty\to\frac{x}{y}+z=\frac{x+zy}{y}.$
Laws for $-$	$(-x)y=-(xy).$
	$(-\frac{1}{x})=\frac{1}{-x}.$
Non triviality	$\infty\neq 0.$

or even the special cases:

Laws of $\frac{1}{\cdot}$	$\frac{1}{\infty}=0$
	$\frac{1}{0\cdot y}=\infty\cdot\frac{1}{y}.$
	$\frac{x}{0}+z=\frac{x+z\cdot 0}{0}.$
Axioms of $-$	$-\bot=\bot.$
	$-\infty=\infty.$

Lemma 5 *If* R *is an integral domain, its quotient wheel is a wheel.*

Proof:
$\frac{a}{b}+\frac{a'}{b'}=\frac{ab'+a'b}{a'b'}=\frac{a'}{b'}+\frac{a}{b}.$
$\frac{a}{b}\cdot\frac{a'}{b'}=\frac{aa'}{bb'}=\frac{a}{b}+\frac{a'}{b'}.$
$(\frac{a}{b}+\frac{c}{d})+\frac{e}{f}=\frac{ad+bc}{bd}+\frac{e}{f}=\frac{adf+bcf+bde}{bdf}=\frac{a}{b}+\frac{cf+de}{df}=\frac{a}{b}+(\frac{c}{d}+\frac{e}{f}).$
If $\frac{a}{b}\neq\infty$, i.e. $f\neq 0\vee e=0$, then $(\frac{a}{b}+\frac{c}{d})\cdot\frac{e}{f}=\frac{ad+bc}{bd}\cdot\frac{e}{f}=\frac{ade+bce}{bdf}=\frac{adef+bcef}{bdff}=\frac{ae}{bf}+\frac{ce}{df}=\frac{a}{b}\cdot\frac{e}{f}+\frac{c}{d}\cdot\frac{e}{f}.$
The other laws are even more straight forward.

Task 6 *Minimize the number of axioms. (More precisely, the number of symbols for writing a complete axiomatization).*

The proofs for the following lemmata are not optimized yet.

Lemma 7 *Assume* $\mathrm{R}_{\perp}^{\infty}$ *is a wheel as above. For* $x, y \in \mathrm{R}_{\perp}^{\infty}$ *we have*

(a) 0, 1, ∞, $\perp$ *are distinct.*

(b) $\mathrm{R}_{\perp}^{\infty} \setminus \{\perp, \infty\}$ *is a field (note that* $\frac{1}{\cdot}$ *is then partial).*

(c) If $x + y = x$, $x \notin \{\infty, \perp\}$*, then* $y = 0$.

(d) If $x \cdot y = x$, $x \notin \{0, \infty, \perp\}$ *then* $y = 1$.

(e) If $x + y = 0$*, then* $y = -x$.

(f) $x \cdot 0 = 0$*, if* $x \notin \{\infty, \perp\}$.

(g) If $x \cdot y = 1$*, then* $y = \frac{1}{x}$.

(h) $-0 = 0$.

(i) $\frac{1}{1} = 1$.

(j) $x + y = \perp \iff x = \perp \vee y = \perp \vee x = y = \infty$.

(k) If $x + y \in \{0, \perp\}$ *then* $(x + y) \cdot \infty = x \cdot \infty + y \cdot \infty$.
Otherwise $(x + y) \cdot \infty = \infty \neq \perp = x \cdot \infty + y \cdot \infty$.

(l) $(-x)y = -(xy) = x(-y)$

(m) $-(-x) = x$.

(n) $-$ *is injective.*

(o) $\frac{1}{\cdot}$ *is injective.*

(p) $\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{xy}$.

(q) $\frac{1}{\frac{1}{x}} = x$.

(r) $\frac{1}{-x} = -\frac{1}{x}$.

(s) $y \neq \infty \to \frac{x}{y} + w = \frac{x+wy}{y}$

(t) $y, y' \notin \{\infty, \perp\} \to \frac{x}{y} + \frac{x'}{y'} = \frac{xy'+x'y}{yy'}$

Proof:
(a) If $\bot = 1$, then $\forall x . x = x \cdot 1 = x \cdot \bot = \bot$, $0 = 1$, contradiction.
If $\bot = 0$, then $\forall x . x = x + 0 = x + \bot = \bot$, $0 = 1$, contradiction.
If $\bot = \infty$, then $0 = \frac{1}{\infty} = \frac{1}{\bot} = \bot$, contradiction.
If $\infty = 0$, then $1 = 1 + 0 = 1 + \infty = \infty = 0$, contradiction.
If $\infty = 1$, then $\bot = 0 \cdot \infty = 0 \cdot 1 = 0$, contradiction.
$0 \neq 1$.
(b) by (a).
The remaining assertions follow now from (b). But let's do it from scratch:
(c) $y = x + y - x = x - x = 0$.
(d) $y = \frac{1}{x} xy = \frac{1}{x} x = 1$.
(e) If $x \notin \{\infty, \bot\}$, then $y = -x + x + y = -x + x = 0$.
If $x = \bot$, $x + y = \bot \neq 0$.
If $x = \infty$, $x + y \in \{\infty, \bot\}$, $x + y \neq 0$.
(f) $x \cdot 0 + x = x \cdot 0 + x \cdot 1 = x \cdot (0 + 1) = x \cdot 1 = x$, $x \cdot 0 = 0$.
(g) If $x \notin \{0, \infty, \bot\}$, then $y = \frac{1}{x} xy = \frac{1}{x} x = 1$.
If $x = 0$, $xy \in \{0, \bot\}$, $xy \neq 1$.
If $x = \bot$, $x \cdot y = \bot \neq 0$.
If $x = \infty$, $x \cdot y \in \{\infty, \bot\}$, $x \cdot y \neq 0$.
(h): $0 + -0 = 0$, $-0 = 0$.
(i): $\frac{1}{1} \cdot 1 = 1$, $1 = \frac{1}{1}$.
(j): "$\leftarrow$" trivial. "$\rightarrow$": Assume right side is false, left side holds.
If $x = \infty$, then $x + y = \infty \neq \bot$.
Otherwise $y = -x + (x + y) = -x + \bot = \bot$, contradicting the falsity of the right side.
(k): Case $x + y = 0$. Then $x \neq \bot$, $(x + y) \cdot \infty = 0 \cdot \infty = \bot$.
Subcase $x = y = 0$: $x \cdot \infty + y \cdot \infty = \bot$.
Otherwise $x, y \notin \{0, \infty, \bot\}$, $x = -y$, $x \cdot \infty + y \cdot \infty = \infty + \infty = \bot$.
Case $x + y = \bot$. Again $(x + y) \cdot z = \bot$.
Case $x = \bot \vee y = \bot$. Right side is $\bot$.
Otherwise follows $x = y = \infty$, right side is $\bot$.
Case $x + y \notin \{0, \bot\}$. Then $(x + y) \cdot \infty = \infty$.
$x \neq \bot$, $y \neq \bot$.
If $x = 0$, then $x \cdot \infty + y \cdot \infty = \bot$.
If $y = 0$, the assertion follows again.
Otherwise $x \cdot \infty + z \cdot \infty = \infty + \infty = \bot$.
(l) If $x \notin \{\infty, \bot\}$, then $(-x)y + xy = (-x + x)y = 0 \cdot y = 0$, $-(xy) = (-x)y$.
If $x = \infty, \bot$, the assertion is an axiom.

(m) $x = \infty, \perp$: by the axioms. Otherwise: $x + (-x) = 0$, therefore $x = -(-x)$.
(n) Assume $-x = -y$.
Case $x = \perp$. $y \neq \infty$. If $y \neq \perp$, then $0 = y + -y = y + \perp = \perp$, contradiction.
Case $x = \infty$: $y \neq \perp$. If $y \neq \infty$, then $0 = y + -y = \infty$, contradiction.
Case $y = \perp$, $y = \infty$: similar.
Otherwise $x = x + -y + y = x + -x + y = y$. (o) Assume $\frac{1}{x} = \frac{1}{y}$.
If $x = \perp$, then $\frac{1}{y} = \perp$, $y \notin \{0, \infty\}$. If $y \neq \perp$, $1 = y \cdot \frac{1}{y} = \perp$, contradiction.
If $x = 0$, then $y \notin \{\perp, \infty\}$. If $y \neq 0$, then $1 = y \cdot \frac{1}{y} = \infty$, contradiction.
If $y = \perp, 0$, follows again the assertion.
Otherwise $x = x\frac{1}{y}y = x\frac{1}{x}y = y$. (p) $x, y \notin \{0, \infty, \perp\}$: $xy\frac{1}{x}\frac{1}{y} = 1$.
$x = \perp$: $\frac{1}{x} \cdot \frac{1}{y} = \perp = \frac{1}{xy}$.
$y = \perp$ similarly.
Assume $x, y \neq \perp$.
Case $x = 0$, $y \neq \infty$: $\frac{1}{y} \notin \{0, \perp\}$. $\frac{1}{xy} = \frac{1}{0} = \infty$, $\frac{1}{x}\frac{1}{y} = \infty\frac{1}{y} = \infty$.
Case $x = 0$, $y = \infty$: both sides are $\perp$.
Case $x = \infty$, $y = 0$: similarly.
Case $x = \infty$, $y \neq 0$: $\frac{1}{xy} = 0 = \frac{1}{x}\frac{1}{y}$.
Case $y = \infty$: similarly.
(q): $x = 0, \infty, \perp$: trivial. Otherwise $\frac{1}{x} \cdot x = 1$, $\frac{1}{\frac{1}{x}} = x$.
(r): The cases $x = 0, \infty, \perp$ can be easily verified. Otherwise we have $(-x) \cdot (-\frac{1}{x}) = -(-(x \cdot \frac{1}{x})) = x$, and the assertion.
(s): If $x = \perp$ or $y = \perp$ or $w = \perp$, both sides are $\perp$. Assume $x, y, w \neq \perp$.
Case $x = y = 0$: $wy \in \{0, \perp\}$, left side is $\perp$, right side is $\frac{0}{0}$ or $\frac{\perp}{0}$, therefore $\perp$.
Case $y = 0$, $w = \infty$: both sides $\perp$.
Case $y = 0$, $w \neq \infty$: both sides are $\perp$, if $x = 0$ and ∞ otherwise.
Otherwise: $y \neq 0, \infty, \perp$, $\frac{x}{y} + w = \frac{1}{y} \cdot y \cdot (\frac{x}{y} + w) = \frac{1}{y} \cdot (x + wy)$.
(t): By (s)

Lemma 8 *Let $a, b, c, d, e, f, r, s \in \mathrm{R}_\perp^\infty \setminus \{\infty, \perp\}$ (as well with indices and accents), $\mathrm{R}_\perp^\infty$ be a wheel.*

(a) Every element of $\mathrm{R}_\perp^\infty$ can be written as $\frac{x}{y}$ with $x, y \neq \perp, \infty$.

(b) $\frac{a}{b} = \frac{a'}{b'} \iff \exists r \notin \{0, \infty, \perp\}.(ar, br) = (a'r', b'r')$.

(c) $\frac{a}{b} + \frac{a'}{b'} = \frac{ab' + a'b}{bb'}$.

(d) $-\frac{a}{b} = \frac{-a}{b}$.

(e) $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$.

(f) $\frac{1}{\frac{a}{b}} = \frac{b}{a}$.

Proof:
(a) $\infty = \frac{1}{0}$, $\bot = \frac{0}{0}$. Otherwise $x = \frac{x}{1}$.
(b) Assume $\frac{a}{b} = \frac{a'}{b'}$. Case $b, b' \neq 0$: $ab' = \frac{a}{b}bb' = \frac{a'}{b'} = a'b$, choose $r = b'$, $r' = b$. If $b = 0 \vee b' = 0$, then $\frac{a'}{b'} = \frac{a}{b} \in \{\infty, \bot\}$, $b = b' = 0$. Further $a = 0 \iff \frac{a}{b} = \bot \iff a' = 0$. Choose, if $a = 0$, $r, r' = 1$ otherwise $r := a'$, $r' := a$.
In the other direction it suffices to show $\frac{a}{b} = \frac{ar}{br}$ if $a, b, r \notin \{\infty, \bot\}$, $r \neq 0$. If $b = 0$, $br = 0$, further $a = 0 \iff ar = 0$ (multiplb with $\frac{1}{r}$), $\frac{a}{b} = \frac{ar}{br}$. Otherwise $\frac{ar}{br}b = \frac{1}{r}rb\frac{ar}{br} = ar\frac{1}{r} = a(r\frac{1}{r}) = a$, $\frac{a}{b} = a\frac{1}{b} = \frac{ar}{br}$.
(c), (d), (e), (f): Immediate by the properties proved before.

References

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