A Model for a type theory with Mahlo Universe

Anton Setzer *

January 14, 1996

Abstract

We present a type theory TTM, extending Martin-Löf Type Theory by adding one Mahlo universe V, a universe being the type theoretic analogue of one recursive Mahlo ordinal. A model, formulated in a Kripke-Platek style set theory KPM^+, is given and we show that the proof theoretical strength of TTM is $\leq |KPM^+| = \psi_{\Omega_1}(\Omega_{M+\omega})$. By [Se96a], this bound is sharp.

1 Introduction

M is recursive Mahlo, iff M is admissible and every M-recursive closed unbounded subset of M contains an admissible ordinal. Equivalent to this is, that M is admissible and for all Δ_0 formulas $\phi(x, y, \vec{z})$, and all $\vec{z} \in L_M$ such that $\forall x \in L_M.\exists y \in L_M.\phi(x, y, \vec{z})$ there exists an admissible $\beta \in L_M$, which is admissible and $\forall x \in L_\beta \exists y \in L_\beta.\phi(x, y, \vec{z})$. One can easily see, that M is recursive inaccessible and that the β above can always be chosen to be recursive inaccessible.
Now one universe in Martin-Löf Type Theory is the least fixed point of an operator, which includes in the presence of the W-type the step to the next admissible. Since, using the W-type we can construct from the universe codes for a least universe, we can regard the universe as the least fixed point. We can construct this therefore precisely by iterating the operator up to the first recursive inaccessible. Proof theoretical analysis on the other hand shows, that we reach using universe and W-type the first recursive inaccessible I in the sense that we can show transfinite induction up to $\psi_{\Omega_1}(I)$ (in fact up to $\psi_{\Omega_1}(\Omega_{I+\omega})$, see [Se95] or [Se93] and the related results in [GR94]). Therefore, universes in type theory correspond to recursive inaccessibles in set theory.

*Institut für Informatik, Universität München, Theresienstr. 39, D-80333 München, Germany, email: `setzer@rz.mathematik.uni-muenchen.de`

Further, the universe and its Tarski-operator T have to be defined simultaneously, therefore the full universe is not only U, but $\Sigma x \in U.T(x) \to U$.

If we put this together, we get the following formulation: V is a Mahlo-universe (together with Tarski-operator T), if V is a universe, and for every function f : $(\Sigma x \in V.T(x) \to V) \to (\Sigma x \in V.T(x) \to V)$ there exists an element $u_f : V$, and a decoding function $s_f : T(u_f) \to V$, such that (with $U_f := T(u_f)$) the following properties hold:

- U_f is a sub-universe of V, i.e. it is closed under ordinary type constructions. For instance, if $a : U_f$ and $x : T(s_f(a)) \Rightarrow b : U_f$ then $\sigma_f x \in a.b : U_f$, and we have $s_f(\sigma_f x \in a.b) = \sigma_V x \in s_f(a).s_f(b) : V$, $T(\sigma_V x \in a.b) = \Sigma x \in T(a).T(b)$.
- There is the restriction of f to U_f: $Res_f : (\Sigma x \in U_f.T(s_f(x)) \to U_f) \to (\Sigma x \in U_f.T(s_f(x)) \to U_f)$ and with $\iota_f : (\Sigma x \in U_f.T(s_f(x)) \to U_f) \to (\Sigma x \in V.T(x) \to V)$, $\iota_f(p(r,t)) := p(s_f(r), \lambda x.s_f(tx))$ we have $\iota_f \circ Res_f = f \circ \iota_f$.

In order, to make this definition possible in Logical Frameworks, we have to split the function f into two functions $g : V \to (T(x) \to V) \to V$ and $h : \Pi x : V.\Pi y : (T(x) \to V).T(gxy) \to V$. Further we have to define first a type $U_{g,h}$ inductively and then define $T(u_{g,h}) := U_{g,h}$. We give the resulting theory preliminarily the name TTM, type theory with Mahlo universe.

In this article, we will present a model for this type theory which is an extension of the model given in [Se93] and [Se96b]. This model will show, that the proof theoretical strength of TTM is $\leq |KPM^+| \leq \psi_{\Omega_1}\Omega_{M+\omega}$. In [Se96a] we show, that this bound is in fact sharp by giving a series of well ordering proofs, showing transfinite induction up to $\psi_{\Omega_1}(\Omega_{M+n})$ for $n \in \omega$. Therefore the resulting theory has exactly the desired strength.

There is **related work** by E. Griffor and M. Rathjen on the Mahlo universe, which treats other aspects of it. There will be probably soon a draft available.

The idea, to work with $\Sigma x \in U.T(x) \to U$ instead of U goes back to E. Palmgren, who told this to the author in the Oberwolfach Meeting in Mathematical Logic 1995. E. Palmgren's studies of the super universe [Pa91] was a first major step towards the Mahlo universe.

2 The Type Theory with one Mahlo Universe

Definition 2.1 *(a) We have the symbols of Martin-Löf Type Theory as in [Se93], definition 2.1, with the following modifications: we remove the term-constructors* $\underline{n}$, $\underline{n}_k$, $\widetilde{+}$, π, σ, w, $\widetilde{i}$ *and the type constructor* U*; we add the term-constructors together with their arity* $\underline{n}_V$ *(0),* $\underline{n}_{k,V}$ *(0) (where* $k \in \omega$*),* $\widetilde{+}_V$ *(2),* π_V *(2),* σ_V *(2),* w_V *(2),* $\widetilde{i}_V$ *(3),* $\underline{n}$ *(2),* $\underline{n}_k$ *(2) (where* $k \in \omega$*),* $\widetilde{+}$ *(4),* π *(4),*

σ (4), w (4), $\widetilde{i}$ (5), u (2), s (3), $Res0$ (4), $Res1$ (5), and the type constructors V (1), U (2).

(b) *The b-objects are defined as in [Se93] definition 2.2 (a). We write $+$, $\widetilde{+}_V$ infix (that is $(a+b)$ for $+(a,b)$) $r\widetilde{+}_{f,g}s$ for $\widetilde{+}fgrs$, $(x)t$ for $\lambda x.t$, $(x,y)t$ for $(x)(y)t$, $(x,y,z)t$ for $(x)(y,z)t$, $\underline{n}_{f,g}$ for $\underline{n}fg$, $\underline{n}_{k,f,g}$ for $\underline{n}_k fg$, $\sigma_{f,g}(r,s)$ for $\sigma(f,g,r,s)$, similarly for π, w, $\widetilde{i}$, $Res0$, $Res1$, u, s and if $S \in \{\Sigma,\ \Pi,\ W,\ \sigma_V,\ \pi_V,\ w_V,\ \sigma_{f,g},\ \pi_{f,g},\ w_{f,g}\}$, $Sx \in s.t := S(s,(x)t)$. Further we write rs for $Ap(r,s)$.
The conventions about omitting brackets, the definition of simultaneous substitution (however we write $t[x_1 := s_1,\dots,x_n := s_n]$ instead of $t[x_1/s_1,\dots,x_n/s_n]$ to avoid confusion) and α-conversion are as in [Se93], definition 2.2.*

(c) *The set of g-objects (for generalised objects) is inductively defined as: variables x are g-terms; if $n < k$, $n,k \in \mathbb{N}$, then n_k is a g-term; if $k \in \mathbb{N}$, then $\underline{n}_k$ is a g-term; if r,s,t,s_0,t_0 are g-terms, $x,x',y,z,u,u',v,v',v'' \in Var_{ML}$, $f := (u,v)s_0$, $g := (u',v',v'')t_0$, then 0, $\underline{r}$, $\underline{n}$, Sr, $\lambda x.r$, $p(r,s)$, $sup(r,s)$, $i(r)$, $j(r)$, $P(r,s,(x,y)t)$, $Ap(r,s)$, $p_0(r)$, $p_1(r)$, $R(r,(x,y,z)s)$, $D(r,(x)s,(x')t)$, $\pi_V x \in r.s$, $\sigma_V x \in r.s$, $w_V x \in r.s$, $r\widetilde{+}_V s$, $\widetilde{i}_V(r,s,t)$, $\pi_{f,g}x \in r.s$, $\sigma_{f,g}x \in r.s$, $w_{f,g}x \in r.s$, $r\widetilde{+}_{f,g}s$, $\widetilde{i}_{f,g}(r,s,t)$, $Res0_{f,g}(r,(x)s)$, $Res1_{f,g}(r,(x)s,t)$, $s_{f,g}(r)$, $u_{f,g}$ are g-terms; if $n \in \mathbb{N}$ and r, $s_1,\dots,s_n$ are g-terms, then $C_n(r,s_1,\dots,s_n)$ is a g-term. Let $Term_{Cl}$ be the set of closed g-terms.*

(d) *The g-types are N_k ($k \in \omega$), N, V and if A,B are g-types, $x,y,z,u,v,u' \in Var_{ML}$, r,s,s_0,t_0 g-terms, $f := (y,z)s_0$, $g := (u,v,u')t_0$, then $\Pi x \in A.B$, $\Sigma x \in A.B$, $Wx \in A.B$, $A+B$, $I(A,r,s)$, $T(r)$, $U_{f,g}$ are g-types.*

(e) *The g-context-pieces, g-contexts, g-statements and $s[\vec{x} := \vec{t}]$ are defined as in [Se93]. We treat here the usual statements $A:$ type and $s:A$ as abbreviations: $A:\ type :\equiv A = A$, $s : A :\equiv s = s : A$*

Definition 2.2 *We define Type Theory with one Mahlo-universe TTM similarly as the definition of $ML_1^eW_T$ in [Se93] definition 2.5, inductively defining $TTM \vdash \Gamma \Rightarrow \Theta$. We only remove all the rules for the Universe and replace them by the following rules for the Mahlo-universe V:*

Type introduction rules for V

$$(V_I)\quad V = V \qquad\qquad (T_I)\ \frac{r = r' : V}{T(r) = T(r')}$$

$$(\pi_V)_I\ \frac{r = r' : V \qquad x : T(r) \Rightarrow s = s' : V}{\pi_V x \in r.s = \pi_V x \in r'.s' : V}$$

$$((\pi_V)_=)\ \frac{r : V \qquad x : T(r) \Rightarrow s : V}{T(\pi_V x \in r.s) = \Pi x \in T(r).T(s)}$$

Similar rules for N, N_k, Σ, W, $+$ and I

$$(u_I)\ \frac{\begin{array}{c} x : V, y : (T(x) \to V) \Rightarrow s_0 = s_0' : V \\ x : V, y : (T(x) \to V), z : T(s_0) \Rightarrow t_0 = t_0' : V \end{array}}{u_{(x,y)s_0,(x,y,z)t_0} = u_{(x,y)s_0',(x,y,z)t_0'} : V}$$

$$(u_=)\ \frac{x : V, y : (T(x) \to V) \Rightarrow s_0 : V \qquad x : V, y : (T(x) \to V), z : T(s_0) \Rightarrow t_0 : V}{T(u_{(x,y)s_0,(x,y,z)t_0}) = U_{(x,y)s_0,(x,y,z)t_0}}$$

Rules for U

$$(U_T)\ \frac{\begin{array}{c} x : V, y : (T(x) \to V) \Rightarrow s_0 = s_0' : V \\ x : V, y : (T(x) \to V), z : T(s_0) \Rightarrow t_0 = t_0' : V \end{array}}{U_{(x,y)s_0,(x,y,z)t_0} = U_{(x,y)s_0',(x,y,z)t_0'}}$$

$$(s_I)\ \frac{\begin{array}{c} r = r' : U_{f,g} \qquad x : V, y : (T(x) \to V) \Rightarrow s_0 = s_0' : V \\ x : V, y : (T(x) \to V), z : T(s_0) \Rightarrow t_0 = t_0' : V \end{array}}{s_{(x,y)s_0,(x,y,z)t_0}(r) = s_{(x,y)s_0,(x,y,z)t_0}(r') : V}$$

Introduction-Rules for U

In the following in all of the introduction-rules we have additional assumptions (rules with subscript I) $x : V, y : (T(x) \to V) \Rightarrow s_0 : V$ and $x : V, y : (T(x) \to V), z : T(s_0) \Rightarrow t_0 : V$;
and in all of the equality-rules (rules with subscript $=$) additional assumptions $x : V, y : (T(x) \to V) \Rightarrow s_0 = s_0' : V$ and $x : V, y : (T(x) \to V), z : T(s_0) \Rightarrow t_0 = t_0' : V$.
Let $f := (x,y)s_0$, $f' := (x,y)s_0'$, $g := (x,y,z)t_0$, $g' := (x,y,z)t_0'$.

$$(Res0_I)\ \frac{r = r' : U_{f,g} \qquad u : T(s_{f,g}(r)) \Rightarrow s = s' : U_{f,g}}{Res0_{f,g}(r,(u)s) = Res0_{f',g'}(r',(u)s') : U_{f,g}}$$

$$(Res0_=)\ \frac{r : U_{f,g} \qquad u : T(s_{f,g}(r)) \Rightarrow s : U_{f,g}}{s_{f,g}(Res0_{f,g}(r,(u)s)) = s_0[x := s_{f,g}(r), y := \lambda u.s_{f,g}(s)] : V}$$

$$(Res1_I)\ \frac{\begin{array}{c} u : T(s_{f,g}(r)) \Rightarrow s = s' : U_{f,g} \qquad t = t' : T(s_{f,g}(Res0_{f,g}(r,(x)s))) \\ r = r' : U_{f,g} \end{array}}{Res1_{f,g}(r,(u)s,t) = Res1_{f',g'}(r',(u)s',t') : U_{f,g}}$$

$$(Res1_{=})\ \frac{r : U_{f,g} \qquad u : T(s_{f,g}(r)) \Rightarrow s : U_{f,g} \qquad t : T(s_{f,g}(Res0_{f,g}(r,(u)s)))}{s_{f,g}(Res1_{f,g}(r,(u)s,t)) = t_0[x := s_{f,g}(r), y := \lambda u.s_{f,g}(s), z := t] : V}$$

$$(\pi_{f,g})_I\ \frac{r = r' : U_{f,g} \qquad x : T(s_{f,g}(r)) \Rightarrow s = s' : U_{f,g}}{\pi_{f,g}x \in r.s = \pi_{f',g'}x \in r'.s' : U_{f,g}}$$

$$(\pi_{f,g})_{=}\ \frac{r : U_{f,g} \qquad x : T(s_{f,g}(r)) \Rightarrow s : U_{f,g}}{s_{f,g}(\pi_{f,g}x \in r.s) = \pi_V x \in s_{f,g}(r).s_{f,g}(s) : V}$$

Similar rules for N, N_k, Σ, W, $+$ and I

3 Definition of KPM^+

Definition 3.1 *(a) The language of KPM^+ is defined as the language of KPi^+ in [Se93] but by adding one constant ad_m (the Mahlo admissible). We will use all the abbreviations of [Se93].*

(b) The axiom schemes for KPM^+ are defined as for KPi^+ in [Se93] (in the extended language) except that we replace the axiom schemes $(+_n)$ by the following:

$$\begin{array}{ll} (Mahlo)^{ad_M} & \forall \vec{z} \in ad_M.((\forall x \in ad_M.\exists y \in ad_M.\phi(x,y,\vec{z})) \rightarrow \\ & \quad \exists w \in ad_M(Ad(w) \wedge \forall x \in w.\exists y \in w.\phi(x,y,\vec{z}))) \\ & \qquad (\phi \text{ a } \Delta_0\text{-formula}) \\ (Ad(ad_M)) & Ad(ad_M) \\ (+_n) & \exists x_1,\ldots,x_n.Ad(x_1) \wedge \cdots \wedge Ad(x_n) \wedge ad_M \in x_1 \wedge x_1 \in x_2 \wedge \cdots \\ & \qquad \wedge x_{n-1} \in x_n. \end{array}$$

(c) KPM^+ is the theory (Ext) + (Found) + (Pair) + (Union) + (Δ_0-sep) + (Δ_0-coll) + (inf) + (Ad.1 - 3) + $(Mahlo)^{ad_M}$ + $(Ad(ad_M))$+ $\{(+_n)|n \in \omega\}$. So KPM^+ is a theory, which guarantees the existence of one recursive Mahlo and of finitely many admissibles above it.

(d) We define L_α, $func(f)$ (for f is a function) $dom(f)$, $rng(f)$ (for domain and range of a function) and Ord (the class of ordinals) as usual. $M := \bigcup_{\alpha \in Ord \cap ad_M} \alpha$. In the following α, β always denote ordinals. $Inacc(\alpha) := \alpha \in Ord \wedge Ad(L_\alpha) \wedge \forall x \in L_\alpha \exists y \in L_\alpha.(Ad(y) \wedge x \in y)$ (α is an inaccessible).

(e) We define $\alpha(a)$, $ad(u)$, $\alpha^+(u)$, Ad_1, Ad_2 as in [Se96b].

(f) In the following in all definitions and lemmata, where we don't mention which part can be done in KPM^+, the statement and definition can be carried out in KPM^+.

4 Interpretation of terms and types

As in [Se96b], definition 4.1 we define the meaning of a_0-extended g-terms, -types, b-objects.

Definition 4.1 *(a) The introductory term constructors are the term constructors* i, j, p, 0, S, $\underline{r}$, sup, $\underline{n}_V$, $\underline{n}_{k,V}$ $(k \in \omega)$, $\tilde{+}_V$, π_V, σ_V, w_V, $\tilde{i}_V$, u.

(b) Let $\rightarrow_{red,imm_{a_0}}$ *or short* $\rightarrow_{red,imm}$ *be the relation between closed* a_0*-extended g-terms, having the clauses for* p_0, p_1, Ap, C_n, D, P, R, A_i *as in [Se93], definition 5.1 (c), additionally:*

$\underline{n}_{f,g} \rightarrow_{red,imm} \underline{n}_V$ $\qquad$ $\underline{n}_{k,f,g}(r,s) \rightarrow_{red,imm} \underline{n}_{k,V}$
$s_{f,g}(r) \rightarrow_{red,imm} r$ $\qquad$ $\sigma_{f,g}(r,s) \rightarrow_{red,imm} \sigma_V(r,s)$
$\pi_{f,g}(r,s) \rightarrow_{red,imm} \pi_V(r,s)$ $\qquad$ $w_{f,g}(r,s) \rightarrow_{red,imm} w_V(r,s)$
$r \tilde{+}_{f,g} s \rightarrow_{red,imm} r \tilde{+}_V s$ $\qquad$ $\tilde{i}_{f,g}(r,s,t) \rightarrow_{red,imm} \tilde{i}_V(r,s,t)$

$Res0_{(x,y)s_0,(x',y',z')t_0}(r,(u)s) \rightarrow_{red,imm} s_0[x := r, y := \lambda u.s]$
$Res1_{(x,y)s_0,(x',y',z')t_0}(r,(u)s,t) \rightarrow_{red,imm} t_0[x' := r, y' := \lambda u.s, z' := t]$

(c) $Term_{nf}$, t^{red}, $t \rightarrow_{red,imm} t'$ *are defined as in [Se93] definition 5.1*

Lemma 4.2 *Lemma 5.3 of [Se93], with* KPi^+ *replaced by* KPM^+*, holds.*

Definition 4.3 *(a) We define* F^α, $Compl(u)$, N_k^{basis}, N_k^{**}, N^{basis}, N_k^{**}, Π^{basis}, Π^*, Σ^{basis}, Σ^*, λ^*, F_W^{basis}, F_W, W^*, $+^{basis}$, $+^*$, i^{basis}, i^* *as in [Se96b].*

(b) $u \rightarrow^* v := \Pi^{basis}(u, \lambda^*(v))$.

(c) For A *being a* Σ*-function,* $n_j \in \omega$, $i \in \omega$ *we define* $A[\vec{x} := \vec{n}]$ *as in [Se93], definition 5.7 (where it is written as* $A[\vec{x}/\vec{n}]$*).*

(d) In the situation of (c), let $(z_i)A$ *be the* Σ*-function with the same arguments as* A *except* u_i *i.e.* $((z_i)A)[\vec{x} := \vec{n}] = \{< u, A[z_i := u, \vec{x} := \vec{n}] > | u \in Term_{Cl}\}$.

(e) Overloading a little bit the notation of (c), we define for $z \in Var_{ML}$, t *a set (which should be a function),* $r \in Term$, $(z)t(r) := \{< s, t(r[z := s]) > \in Term_{Cl} \times TC(t) | r[z := s] \in dom(t)\}$.

(f) We define the interpretation of g-types A *as in [Se96b], except that we remove the definition of* U^*, *and* $T(t)^*$ *and add instead:*
$lev(V) := 1$, $V^* := \hat{V}^M$,
$lev(T(t)) := 0$, $(T(t))^*[\vec{x} := \vec{s}] := \hat{T}^M(t[\vec{x} := \vec{s}])$,
$lev(U_{f,g}) := 0$, $(U_{f,g})^* := \hat{T}^M(u_{f,g})$,
where $\hat{V}^\alpha$, $\hat{T}^\alpha$ *are defined in the next definition.*
For finitely many g-types this definition can be done in KPM^+.

Definition 4.4 *(a)* $Complv(v) := \{< r, r' >\in Term_{Cl} \times Term_{Cl}|$
$\exists s, s' \in Term_{nf}.r \rightarrow_{red} s \wedge r' \rightarrow_{red} s' \wedge < s, s' >\in v\}$.

$Complt(tv) := \{< s, v >\in Term_{Cl} \times TC(tv)|$
$\exists s' \in Term_{Cl}.s \rightarrow_{red} s' \wedge < s', v >\in tv\}$.

(b) $\tilde{V}_\alpha(fv, tv) := Complv(\tilde{V}^{basis}_\alpha(fv, tv))$, $\tilde{T}_\alpha(fv, tv) := Complt(\tilde{T}^{basis}_\alpha(fv, tv))$, *where*

$$
\begin{aligned}
&\tilde{V}^{basis}_\alpha(fv, tv) := \{< \underline{n}_V, \underline{n}_V >\}\\
\cup\ &\{< \underline{n}_{k,V}, \underline{n}_{k,V} > |k \in \omega\}\\
\cup\ &\{< \pi_V x \in r.s, \pi_V x' \in r'.s' >\in Term_{nf} \times Term_{nf}|\\
&\quad \phi(r, x, s, r', x', s', fv(\alpha), tv)\}\\
\cup\ &\{< \sigma_V x \in r.s, \sigma_V x' \in r'.s' >\in Term_{nf} \times Term_{nf}|\\
&\quad \phi(r, x, s, r', x', s', fv(\alpha), tv)\}\\
\cup\ &\{< w_V x \in r.s, w_V x' \in r'.s' >\in Term_{nf} \times Term_{nf}|\\
&\quad \phi(r, x, s, r', x', s', fv(\alpha), tv)\}\\
\cup\ &\{< r \tilde{+}_V s, r' \tilde{+}_V s' >\in Term_{nf} \times Term_{nf}|\\
&\quad | < r, r' >, < s, s' >\in fv(\alpha)\}\\
\cup\ &\{< \tilde{i}_V(r, s, t), \tilde{i}_V(r', s', t') >\in Term_{nf} \times Term_{nf}|\\
&\quad | < r, r' >\in fv(\alpha) \wedge < s, s' >, < t, t' >\in tv(r)\}\\
\cup\ &\{< u_{(x,y)s,(x_0,y_0,z)t}, u_{(x',y')s',(x'_0,y'_0,z')t'} >\in Term_{nf} \times Term_{nf}|\\
&\quad |\psi(x, y, s, x_0, y_0, z, t, x', y', s', x'_0, y'_0, z', t', \beta, \alpha + 1, fv, tv)\},
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{T}^{basis}_\alpha(fv, tv) := \{< \underline{n}_V, N^{**} >\}\\
\cup\ &\{< \underline{n}_{k,V}, N^{**}_k > |k \in \omega\}\\
\cup\ &\{< \pi_V x \in r.s, \Pi^*(tv(r), (x)tv(s)) >\in Term_{nf} \times ad(fv \cup tv)|\\
&\quad \phi(r, x, s, r, x, s, fv(\alpha), tv)\}\\
\cup\ &\{< \sigma_V x \in r.s, \Sigma^*(tv(r), (x)tv(s)) >\in Term_{nf} \times ad(fv \cup tv)|\\
&\quad \phi(r, x, s, r, x, s, fv(\alpha), tv)\}\\
\cup\ &\{< w_V x \in r.s, W^*(tv(r), (x)tv(s)), \alpha^+(fv \cup tv) >\\
&\quad \in Term_{nf} \times ad(ad(fv \cup tv))|\phi(r, x, s, r, x, s, fv(\alpha), tv)\}\\
\cup\ &\{< r \tilde{+}_V s, +^*(tv(r), tv(s)) >\in Term_{Cl} \times ad(fv \cup tv)|\\
&\quad < r, r >, < s, s >\in fv(\alpha)\}\\
\cup\ &\{< \tilde{i}_V(r, s, t), I^*(tv(r), s, t) >\in ad(fv \cup tv)|\\
&\quad < r, r >\in fv(\alpha) \wedge < s, s >, < t, t >\in tv(r)\}\\
\cup\ &\{< u_{(x,y)s,(x_0,y_0,z)t}, fv(\beta) >\in Term_{nf} \times ad(fv \cup tv)|\\
&\quad \psi(x, y, s, x_0, y_0, z, t, x, y, s, x_0, y_0, z, t, \beta, \alpha + 1, fv, tv)\}
\end{aligned}
$$

$$
\begin{aligned}
&\phi(r, x, s, r', x', s', v, tv) :=\\
&\quad < r, r' >\in v \wedge (\forall < t, t' >\in tv(r). < s[x := t], s'[x' := t'] >\in v)\\
&\psi(x, y, s, x_0, y_0, z, t, x', y', s', x'_0, y'_0, z', t', \beta, \alpha, fv, tv) :=\\
&\quad \psi'(x, y, s, x_0, y_0, z, t, x', y', s', x'_0, y'_0, z', t', \beta, \alpha, fv, tv) \wedge\\
&\quad \forall \beta' < \beta \neg \psi'(x, y, s, x_0, y_0, z, t, x', y', s', x'_0, y'_0, z', t', \beta', \alpha, fv, tv),
\end{aligned}
$$

and

$\psi'(x, y, s, x_0, y_0, z, t, x', y', s', x'_0, y'_0, z', t', \beta, \alpha, fv, tv) :=$

$Inacc(L_\beta) \wedge \beta < \alpha \in Ord \wedge$
$(\forall <r, r'> \in fv(\beta). \forall <r_0, r'_0> \in tv(r) \rightarrow^* fv(\beta).$
$(<s[x := r, y := r_0], s'[x' := r', y' := r'_0]> \in fv(\beta) \wedge$
$\forall <s_0, s'_0> \in tv(s[x := r, y := r_0]).$
$<t[x := r, y := r_0, z := s_0], t'[x' := r', y' := r'_0, z := s'_0]> \in fv(\beta))).$

(c) *By simultaneous recursion on* $\alpha \in Ord$ *we define* $\hat{V}^\alpha, \hat{T}^\alpha$*:*
$\hat{V}^0 := \hat{T}^0 := \emptyset$.
$\hat{V}^{\alpha+1} := \tilde{V}_\alpha(\hat{V}|\alpha, \hat{T}^\alpha)$, $\hat{T}^{\alpha+1} := \tilde{T}_\alpha(\hat{V}|\alpha, \hat{T}^\alpha)$,
where $\hat{V}|\alpha := \{<\gamma, \hat{V}^\gamma> | \gamma \leq \alpha\}$.
$\hat{V}^\lambda := \bigcup_{\alpha<\lambda} \hat{V}^\alpha$, $\hat{T}^\lambda := \bigcup_{\alpha<\lambda} \hat{T}^\alpha$ *for* λ *limit ordinal.*

We verify that similar properties as in 6.1 - 6.17 of [Se93] hold and assume the same definitions as there. Additionally we have

Lemma 4.5 $\forall \alpha \in M. U{\restriction}(\alpha+1), T^\alpha \in L_M$.

5 Main Lemma

Lemma 5.1 *(***Main lemma***)*
Let Γ, Δ *be g-context-pieces,* $x, x_i \in Var_{ML}$*,* A_i, A, B *g-types,* t, t' *g-terms,* θ *a g-judgement. Assume* $\Gamma = x_1 : A_1, \ldots, x_n : A_n$.

(a) *If* $TTM \vdash \Gamma \Rightarrow t = t' : A$*, then*

  (i) $KPM^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(t = t' : A)^*[\vec{x}/\vec{r}; \vec{s}]$.

  (ii) $KPM^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A = A)^*[\vec{x}/\vec{r}; \vec{s}]$.

(b) *If* $TTM \vdash \Gamma \Rightarrow A = A'$*, then*

$$KPM^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A = A')^*[\vec{x}/\vec{r}; \vec{s}].$$

(c) *If* $TTM \vdash \Gamma, x : A, \Delta \Rightarrow \theta$*, then*

$$KPM^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A = A)^*[\vec{x}/\vec{r}; \vec{s}].$$

Proof: As lemma 6.18 in [Se93], with changes for the new rules. Because of lack of space, we only give hints for some rules and treat the rule (u_I) carefully. For simplicity we assume for this rule that the context Γ is empty.
Assume $\alpha < M$. If $<r, r'> \in \hat{V}^\alpha$, $<s, s'> \in \hat{T}^\alpha(s) \rightarrow^* \hat{V}^\alpha$, then $<r, r'> \in V^*$, $<s, s'> \in (T(s) \rightarrow V)^*$, by IH therefore $<s_0[x := r, y := s], s'_0[x := r', y := s']> \in V^\beta$ for some $\alpha < \beta < M$. By $(\Delta_0 - Coll)^M$ there exists $\alpha < h_0(\alpha) < M$ such that all β can be chosen to be less than $h_0(\alpha)$.

If $< t, t' >\in \hat{T}^{h_0(\alpha)}(s_0[x := r, y := s])$, then $< t_0[x := r, y := s, z := t], t'_0[x := r', y := s', z := t'] >\in \hat{V}^\gamma$ for some γ, again by the collection-axiom we can find some $h_0(\alpha) < h_1(\alpha) < M$ such that we can bound the γ as before by $h_1(\alpha)$.
Further we can find $h_1(\alpha) < h_2(\alpha) < M$ such that $Ad(h_2(\alpha))$.
Since M is Mahlo, we can find some $\beta < M$ s.t. $Ad(\beta)$, $\forall\alpha < \beta.h_2(\alpha) < \beta$.
Therefore $Inacc(\beta)$ and β fulfils $\psi'(x, y, s_0, x, y, z, t_0, x, y, s'_0, x, y, z, t'_0, \beta, \beta + 1, \hat{U}|\beta, \hat{T}^\beta)$, therefore with the minimal such β' we have $\psi(x, y, s_0, x, y, z, t_0, x, y, s'_0, x, y, z, t'_0, \beta', \beta + 1, \hat{U}|\beta, \hat{T}^\beta)$, and therefore $< u_{(x,y)s_0,(x,y,z)t_0}, u_{(x,y)s'_0,(x,y,z)t'_0} >\in \hat{V}^{\beta+1} \subset \hat{V}^M$.
The other rules are easy, since $U^*_{(x,y)r,(x,y,z)s} = (T(u_{(x,y)r,(x,y,z)s}))^* = V^\beta$ (if the assumptions as for the rule u_I are fulfilled) for some inaccessible β, therefore has (as in the construction of U^* in [Se93]) the closure properties of a universe, and by our reduction rules we identify π_V and $\pi_{f,g}$, the same for the other type constructors, omit $s_{f,g}$ and evaluate $Res0_{f,g}$, $Res1_{f,g}$.
A more detailed proof will be given in a forthcoming paper.

6 Results

Lemma 6.1 $|KPM^+| \leq \psi_{\Omega_1}(\Omega_{M+\omega})$.

Proof: Adapt [Ra91] or [Bu93] as in similar to the arguments for KPi^+ in the proof of theorem 7.8 in [Se93].

Remark: We write $\leq$ instead of $=$ in 6.1 only, because we haven't seen a well-ordering proof for KPM yet (which could then be lifted to KPM^+) although such a proof probably exists. However the sequence (which is an unnecessary detour) $|TTM| \leq |KPM^+| \leq \psi_{\Omega_1}(\Omega_{M+\omega}) \leq |TTM|$ shows anyway, that $=$ holds.

Lemma 6.2 *Assume ϕ is a Π^1_1-formula. Let $\hat{\phi}$ be the standard interpretation of ϕ in type theory and $\tilde{\phi}$ its interpretation in KPM^+ as in [Se96b] (with the power set of the natural numbers restricted to sets in Ad_1). Then, if $TTM \vdash s : \hat{\phi}$, follows $KPM^+ \vdash \tilde{\phi}$.*

Proof: By modifying the interpretation slightly by taking in the above proof the corresponding definitions and proofs of [Se96b] instead of those in [Se93] and then as in [Se93].

Theorem 6.3 $|TTM| \leq \psi_{\Omega_1}(\Omega_{M+\omega})$.

Proof: as in [Se96b].

References

[Bu92] Buchholz, W.: *A simplified version of local predicativity.* In: Aczel, P. et al. (Eds.): *Proof Theory.* Cambridge University Press, Cambridge, 1992, pp. 115 – 147.

[Bu93] Buchholz, W.: *A note on the ordinal analysis of* KPM in: J. Oikkonen, J. Väänänen (Eds.): *Logic Colloquium'90, Proceedings of the ASL Summer Meeting Helsinki.* Springer Lecture Notes in Logic 3, 1993, pp. 1 - 9.

[GR94] Griffor, E. and Rathjen, M.: *The strength of some Martin-Löf type theories.* Arch. Math. Log. 33, 1994, pp. 347 – 385.

[Pa91] Palmgren, E.: *Transfinite Hierarchies of Universes.* U.U.D.M. Report 1991:7, Uppsala University, Department of Mathematics, Thunbergsvägen 3, S-752 38 Uppsala, Sweden.

[Ra91] Rathjen, M.: *Proof-Theoretical Analysis of* KPM, Arch. math. log. 30, 1991, pp. 377 – 403.

[Se93] Setzer, A.: *Proof theoretical strength of Martin-Löf Type Theory with W-type and one universe.* PhD-thesis, Munich, 1993. available via `http://www.mathematik.uni-muenchen.de/~setzer/articles/weor0.dvi.gz`

[Se95] Setzer, A.: *Well-ordering proofs for Martin-Löf Type Theory with W-type and one universe.* Submitted to Annals of Pure and Applied Logic. Available via `http://www.mathematik.uni-muenchen.de/~setzer/articles/2papdiss.dvi`

[Se96a] Setzer, A.: *A Type Theory for one Mahlo Universe.* A preliminary version is available via `http://www.mathematik.uni-muenchen.de/~setzer/articles/mahlo.dvi.gz`

[Se96b] Setzer, A.: *An upper bound for the proof theoretical strength of Martin-Löf Type Theory with W-type and one Universe.* In preparation. A preliminary version is available via `http://www.mathematik.uni-muenchen.de/~setzer/articles/1papdiss.dvi.gz`