Predicativity of the Mahlo Universe in Type Theory

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Abstract

We provide a constructive, predicative justification of Setzer's Mahlo universe in type theory. Our approach is closely related to Kahle and Setzer's construction of an extended predicative Mahlo universe in Feferman's Explicit Mathematics. However, we work directly in Martin-Löf type theory extended with a Mahlo universe and obtain informal meaning explanations which extend (and slightly modify) those in Martin-Löf's article *Constructive Mathematics and Computer Programming*. We also present mathematical models in set-theoretic metalanguage and explain their relevance to the informal meaning explanations.

Martin-Löf's first published paper on type theory was entitled "An intuitionistic type theory: predicative part" [12]. This theory had an infinite hierarchy of universes. Its proof-theoretic strength was determined to be Γ_0 [5, 6], the limit of predicativity in Feferman and Schütte's sense [20, 9, 8, 4, 18, 17]. In his article *Constructive Mathematics and Computer Programming* [13] Martin-Löf added W-types, and the theory became impredicative in Feferman's and Schütte's sense. Nevertheless, Martin-Löf still considered the theory predicative in an extended sense, since *meaning explanations* were given suggesting how the types and terms of the theory are built up from below. They explain how the objects of the theory are trees that are built by a well-founded process of repeated lazy evaluation of expressions to canonical form.

Martin-Löf type theory was later extended with several higher universe constructions, such as Palmgren's universe operators, the super universe [15], quantifier universes [16], and Setzer's Mahlo universe [19]. All these extensions were intended to be constructive and predicative in the sense of Martin-Löf's meaning explanations. However, the predicativity of the Mahlo universe was not so clear, especially after Palmgren [15] discovered that adding a natural elimination rule for it led to an inconsistency. Maybe Mahlo is a natural limit of Martin-Löf's extended predicativity as we conjectured in our paper on a finite axiomatisation of inductive-recursive definitions [2]?

A universe in type theory is a type closed under all the standard type formers of Martin-Löf type theory, such as $\Pi, \Sigma, 0, 1, 2, \mathrm{N}, \mathrm{W}$, and the identity type I. Universes can either be formulated à la Russell, where an element $A : \mathrm{U}$ is also a type A, or à la Tarski, where an element $a : \mathrm{U}$ is a "name" or "code" of a type A and there is a decoding map T such that $\mathrm{T}\,a = A$.

A super universe is a universe closed under an operator on families of sets that maps a universe $(\mathrm{U}_n, \mathrm{T}_n)$ to the next universe $(\mathrm{U}_{n+1}, \mathrm{T}_{n+1})$ in the hierarchy. One can then form an operator mapping a super universe to the next and form a super^2 universe closed under this operator. This process can be iterated and thus one obtains super^n universes. More generally, one can define universes closed under arbitrary operators on families of sets. A Mahlo universe is a universe that contains all universes generated by family operators. Moreover, the latter are *subuniverses* of the Mahlo universe. One can show that these subuniverses arise as special cases of the inductive-recursive definitions in our theory **IR** [2]. This theory is formulated as an extension of Martin-Löf's logical framework [14], where there is a type Set of "sets" in Martin-Löf's sense: "to know a *set* is to know how the elements of the set are formed and how equal elements are formed", a phrase indicating that sets should be inductively (or inductive-recursively) generated. Therefore we will refer to $\Pi, \Sigma, 0, 1, 2, \mathrm{N}, \mathrm{W}, \mathrm{I}$, etc, as *set formers* rather than type formers, when we work in this version of type theory. (Note that Martin-Löf's notion of "set" is different from the "h-sets" in homotopy type theory.)

Let f be an operator on families of sets split into two components (f_0, f_1) where f_0 returns the index set and f_1 returns the family. Then we can define a subuniverse $\mathrm{U}\, f_0\, f_1 : \mathrm{Set}$ with decoding $\mathrm{T}\, f_0\, f_1 : \mathrm{U}\, f_0\, f_1 \to \mathrm{Set}$ as an instance of an inductive-recursive definition in **IR**. In this way Set encodes Setzer's Mahlo universe [19]. We call it an *external* Mahlo universe to contrast it with the *internal* Mahlo universe that arises if we introduce a set $\mathrm{M} : \mathrm{Set}$ with the Mahlo property. This M goes beyond inductive-recursive definitions in **IR**.

When Martin-Löf extended his meaning explanation to the 1986 version based on a logical framework [14], he did *not* stipulate that "to know a *type* is to know how the objects of the type are formed and how equal objects are formed". (We refer to Martin-Löf's Leiden lectures [10, 11] for a comprehensive account of the philosophical foundations of intuitionistic type theory with the distinction between types and sets.) The type Set is to be understood as "open" to extension with new inductive(-recursively) defined sets when we need them. Hence it is not natural to add an elimination rule for it. In contrast to this, $\mathsf{M} : \mathrm{Set}$ is to be understood as "closed". Nevertheless, as Palmgren showed, adding a natural elimination rule for it leads to an inconsistency. This paradox makes us doubt whether the internal Mahlo universe is a good predicative set according to Martin-Löf's conception.

In spite of this, we shall argue that the Mahlo universe is predicative and constructive by giving Martin-Löf style meaning explanations for it. Our argument can be applied both to the external and internal Mahlo universe, although we only discuss the simpler external version.

We first construct a crude set-theoretic model of logical framework-based type theory with Set as a Mahlo universe and $\mathrm{U}\, f_0\, f_1 : \mathrm{Set}$ with decoding $\mathrm{T}\, f_0\, f_1 : \mathrm{U}\, f_0\, f_1 \to \mathrm{Set}$ as subuniverses. This model is an adaptation of our model of **IR** [2], where type-theoretic function spaces are interpreted as sets of all set-theoretic functions. We work in the classical set theory **ZFC** with a Mahlo cardinal M and an inaccessible cardinal I above it. (Note that we use the term "set" both for sets in Martin-Löf type theory and for sets in the set-theoretic metalanguage. We hope this will not lead to confusion.) We interpret the collection of all types as V_{I} and Set as V_{M}. Let $\mathcal{F}am(V) = \{(X, Y) \mid X \in V, Y : X \to V\}$ be the set of families of sets in V. An operator on families of sets in the type theory is interpreted as a function $f : \mathcal{F}am(\mathrm{V}_{\mathsf{M}}) \to \mathcal{F}am(\mathrm{V}_{\mathsf{M}})$. We then use the Mahlo property of M to show that there is an inaccessible cardinal $\kappa_f < \mathsf{M}$ such that $f : \mathcal{F}am(\mathrm{V}_{\kappa_f}) \to \mathcal{F}am(\mathrm{V}_{\kappa_f})$ and interpret the subuniverses $\mathrm{U}\, f_0\, f_1$ as $\mathcal{U}\, f_0\, f_1 = \mathrm{V}_{\kappa_f}$ à la Russell, that is, the decoding $\mathrm{T}\, f_0\, f_1$ is interpreted as the injection $\mathcal{T}\, f_0\, f_1 : \mathrm{V}_{\kappa_f} \hookrightarrow \mathrm{V}_{\mathsf{M}}$.

We then construct a second "predicative" set-theoretic model where we interpret the inductive-recursively defined type-theoretic subuniverses $(\mathrm{U}\, f_0\, f_1, \mathrm{T}\, f_0\, f_1)$ in set theory in terms of inductively generated graphs $\mathcal{T}\, f_0\, f_1$ with domain $\mathcal{U}\, f_0\, f_1$ in the standard set-theoretic way following Allen [1]. Moreover, in order to interpret Set as an inductively defined set $\mathcal{S}et$ we make use of Kahle and Setzer's extended predicative Mahlo universe in Feferman's theory of Explicit Mathematics [7, 3]. The key idea is that it suffices to require that the family operator f on families of sets is total on families over the subuniverse $\mathcal{U}\, f_0\, f_1$ when we add $\mathcal{U}\, f_0\, f_1$ to $\mathcal{S}et$. Although this may seem impredicative, we show that it results in an inductive definition of $\mathcal{S}et \subseteq \mathrm{V}_{\mathsf{M}}$. Moreover, we show that $\mathcal{T}\, f_0\, f_1 : \mathcal{U}\, f_0\, f_1 \to \mathcal{S}et$.

The final step is to provide Martin-Löf style meaning explanations inspired by the second set-theoretic model. The usual situation in type theory is that the meaning explanation for a type is determined by the formation rule and the introduction rules, and the computation rules for the elimination constant is given by the equality rules. However, in the case of the Mahlo universe this pattern is broken. If we take the formation rule for the subuniverses $\mathrm{U}\, f_0\, f_1$ as a type-checking condition (a *matching condition*), then we get a non-wellfounded type-checking process, because of Palmgren's paradox. (We remark that "type checking" here refers to the matching of canonical terms with canonical types in Martin-Löf's meaning explanations, and not to the type-checking of judgements in intensional type theory, as implemented in proof-assistants.)

Instead we let the second set-theoretic model suggest the type-checking conditions. As an example we give one of the type-checking conditions for the judgement $A : \mathrm{Set}$. If A has canonical form $\mathrm{U}\, f_0\, f_1$, then we check whether

$$f_0\, (\mathrm{T}\, f_0\, f_1 u)\, (\lambda x.\mathrm{T}\, f_0\, f_1(tx)) : \mathrm{Set}$$

in the context $u : \mathrm{U}\, f_0\, f_1, t : \mathrm{T}\, f_0\, f_1\, u \to \mathrm{U}\, f_0\, f_1$, and

$$f_1\, (\mathrm{T}\, f_0\, f_1 u)\, (\lambda x.\mathrm{T}\, f_0\, f_1(tx))\, y : \mathrm{Set}$$

in the context $u : \mathrm{U}\, f_0\, f_1, t : \mathrm{T}\, f_0\, f_1\, u \to \mathrm{U}\, f_0\, f_1, y : f_0\, (\mathrm{T}\, f_0\, f_1 u)\, (\lambda x.\mathrm{T}\, f_0\, f_1(tx))$. We avoid the circularity by only type-checking for arguments in the image of $\mathrm{T}\, f_0\, f_1 : \mathrm{U}\, f_0\, f_1 \to \mathrm{Set}$ and this avoids checking for $\mathrm{U}\, f_0\, f_1 : \mathrm{Set}$. Nevertheless, the formation rule for $\mathrm{U}\, f_0\, f_1$ can be justified on this basis.

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