Implementing Martin-Löf's Meaning Explanations for Intuitionistic Type Theory in Agda

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At the Congress of LMPS in Hannover in 1979 Martin-Löf presented his seminal paper "Constructive Mathematics and Computer Programming" [1]. In this paper there is not only a new version of intuitionistic type theory but it also contained so called "meaning explanations" for this theory. The meaning explanations can be viewed both meta-mathematically, as a certain kind of model construction in the tradition of Kleene realizability, and philosophically, as a "pre-mathematical" discussion of the direct, intuitive semantics of type theory.

In this talk we will discuss both aspects, and present a formalization of meaning explanations in Agda (work in progress). More specifically, we will formalize the syntax and computation rules of a fragment of intuitionistic type theory, in order to make meaning explanations written in natural language more precise.

The computation relation relates an untyped term to its canonical form, where a canonical form is a term the outermost form of which is a constructor. The meaning of the judgements of type theory is then defined in terms of such computation to canonical form. Finally, we prove the correctness of some of the inference rules of intuitionistic type theory. In particular we show that the rules of extensional identity types are validated.

Meaning explanations represent an important contribution to the foundations of intuitionistic logic and constructive mathematics. They clarify the Brouwer-Heyting-Kolmogorov interpretations of the logical constants and also generalize them. They do not only explain the meaning of the logical constants but also provide a conception of what a constructive mathematical object is in general.

From a "pre-mathematical" point of view, the aim of the meaning explanations is to make the correctness of each rule of intuitionistic type theory immediately evident. In "Constructive Mathematics and Computer Programming" Martin-Löf famously finished with the following statement:

> For each of the rules of inference, the reader is asked to make the conclusion evident to himself on the presupposition that he knows the premises. This does not mean that further verbal explanations are of no help in bringing about an understanding of the rules, only that this is not the place for such detailed explanations. But there are also certain limits to what verbal explanations can do when it comes to justifying axioms and rules of inference. In the end, everybody must understand for himself.

Such further informal verbal explanations were provided in Martin-Löf's 1984 book "Intuitionistic Type Theory" [2]. In our formalization in Agda we prove formally the correctness of the inference rules in full detail. This provides rigorous justifications for them, and we find it

interesting to carry out those details. An essential aspect of the work is of course that we use a fragment of Agda far more complex than the theory we are justifying. In particular, we use features such as indexed inductive-recursive definitions and Agda's pattern matching, to make a transparent formalization.

Martin-Löf's meaning explanations provide the perhaps most convincing foundations of (predicative) constructive mathematics. To properly appreciate them one needs to understand both the technical formal meta-mathematical side, and the informal pre-mathematical side. We hope that our work will help in this endeavour.

References

[1] Per Martin-Löf. Constructive mathematics and computer programming. In L. Jonathan Cohen, Jerzy Łoś, Helmut Pfeiffer, and Klaus-Peter Podewski, editors, *Logic, Methodology and Philosophy of Science VI, Proceedings of the Sixth International Congress of Logic, Methodology and Philosophy of Science, Hannover 1979*, volume 104 of *Studies in Logic and the Foundations of Mathematics*, pages 153–175. North-Holland, 1982. doi:10.1016/S0049-237X(09)70189-2.

[2] Per Martin-Löf. *Intuitionistic type theory*, volume 1 of *Studies in Proof Theory*. Bibliopolis, 1984.