Ordinal Systems, Part 2: One Inaccessible

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Abstract. We develop an alternative approach to well-ordering proofs beyond the Bachmann-Howard ordinal using transfinite sequences of ordinal notations and use it in order to carry out well-ordering proofs for σ-ordinal systems. We extend the approach of ordinal systems as an alternative way of presenting ordinal notation systems started in [18] and develop ordinal systems, which have in the limit exactly the strength of Kripke-Platek set theory with one recursivly inaccessible. The upper bound is determined by giving well-ordering proofs, which use the technique of transfinite sequences. We derive from the new approach the traditional approach to well-ordering proofs using distinguished sets. The lower bound is determined by extending the concept of ordinal function generators in [18] to inaccessibles.

1 Introduction

This article is a followup of [18]. In that article we introduced ordinal systems as an alternative way of describing ordinal notation systems which usually make use of collapsing functions. In the simplest case of one single ordinal system, ordinals were always denoted by referring to smaller ones, a principle, which is valid in the case of systems built from extensions of the Veblen function. In order to get beyond the Bachmann Howard ordinal, we used several ordinal systems in parallel, such that the ordinal systems refer to each other in a controlled way. We described, how the combination of ordinal systems was used in order to generate in a well-ordered way the ordinal notation systems in question. This showed that they are well-ordered. We considered and analyzed ordinal systems up to the proof theoretic strength of ID_σ.

In the current article we proceed with this research and present I-ordinal systems: ordinal systems which reach in the limit the strength of KPI, Kripke Platek set theory with one recursively inaccessible I (more precisely, the class of ordinals in KPI has the properties of one recursively inaccessible; proof theoretic studies for variations of Kripke Platek set theory can be found in [9], a formalization of KPI can be found in [9] or looked up as well in [4]). We will then determine their strength as that of KPI. This presentation should help to understand ordinal notation systems of this strength in a better way.

The content of the article is as follows: I-ordinal systems will be introduced in Section 3. In Sections 4 and 5 we will then show that I-ordinal systems are well-ordered and that elementary I-ordinal systems, i.e. I-ordinal systems, the properties of which can be verified in PRA, have order type less than KPI. We are going to present two well-ordering proofs. One, using a new technique, namely transfinite sequences of ordinal notations, and a second in the more traditional way, which uses distinguished sets and can be used for well-ordering proofs in constructive theories. In order to motivate the new technique of using sequences, we introduce it in Section 2 in the context of σ-ordinal systems. In the last section of this article (Section 6), we will extend the notion of ordinal function generators and define ordinal function generators corresponding to I-OS. We will see there that we reach the strength of I-OS. Therefore the strength of elementary I-ordinal systems is determined as being precisely the strength of KPI.

2 A variation of the usual well-ordering proof techniques

The usual technique in proof theory for carrying out well-ordering proofs is the technique of distinguished sets, introduced by Buchholz in [1]. In our previous article [18] some (but not the full power of it) was used and it seems to be still the only technique available for carrying out well-ordering proofs of strength beyond ID_ω in constructive theories. However, we believe that for theories which have substantially stronger strength it is not so easy to understand what is going on in it, and found it very difficult to teach it. We have now developed a variation of it, which we hope is slightly more intuitive. The disadvantage of it is that it can in the current form not be carried out in constructive theories. However this is no harm since from it in a second step we can derive the technique of distinguished sets. We hope our presentation motivates distinguished sets in a better way.

Instead of defining the sets in an abstract way we are going to define by recursion on set theoretic ordinals a transfinite sequence of ordinal notations a_α. Distinguished sets will occur as initial segments of the set of notations generated in such a way. In order to show the well-foundedness of the resulting system, apart from some verifications, which can be essentially be carried out in primitive recursive arithmetic, and the existence of such a sequence, which is clear in the case of Kripke Platek set theory — the use of classical logic is crucial here —, we will need that, when iterated over all ordinals, the resulting sequence is complete, i.e. this sequence can not be extended.

In order to prove the consistency of the theories in question we will need therefore essentially that the class of ordinals is big enough. One can use principles which claim that the class of ordinals fulfills the properties of some recursively large ordinal. Or one can claim instead that the set of ordinals should be unbounded in the sense that we cannot exhaust it by any recursive process, and that therefore such an ordinal must exist. The latter point of view provides an alternative argument for the consistency of the theory in question, alternative to the use of

constructive theories, the consistency of which can be philosophically shown or at least justified. At the moment, the new argument is not as well philosophically developed as the other approach, but we hope to be able to investigate it in the near future. In this article, we only mention the possibility of such an approach. Once we have found such a sequence and carried out the well-ordering proof using it, we can now find the constructive version of the well-ordering proof using distinguished sets by formalizing abstractly the principles according to which the sets $\mathrm{A}_\alpha := \{\mathrm{a}_\beta \mid \mathrm{a}_\beta \downarrow \wedge \beta < \alpha\}$ are defined.
We will apply this in section 4 to I-ordinal systems. In a first step we want to look at σ-ordinal systems in order to introduce the technique. In the case of σ-ordinal systems, the benefits will not be very big, most readers will certainly prefer the old technique. However for stronger systems this technique will motivate well-ordering proofs in a better way.

We need a theory of the strength of ID_σ in which we have ordinals available. Kripke Platek set theory without Δ_0-collection, but extended by natural numbers and the existence of a well-ordering Σ (as new urelemente) of order type σ and for every $a \in \Sigma$ an admissible such that these admissibles are ordered as the corresponding elements of Σ and the collection of predecessors of an $a \in \Sigma$ is a set, which is an element of the ath admissible, should be the correct theory. (The exact formulation of such a theory is not important here since this Section mainly serves to introduce and motivate the new technique). In this section, (set-theoretic) ordinal refers to the set of ordinals below some of the admissibles which were additionally introduced in this theory.
Before defining such sequences we make our life easier by extending the order $<_\xi$ to a global order:

Definition 2.1. (a) $\mathrm{T} := \bigcup_{\xi\in\Sigma} \mathrm{T}_\xi$, $\mathrm{T}_{<\xi} := \bigcup_{\mu<\xi} \mathrm{T}_\mu$, $\mathrm{T}_{\leq\xi} := \bigcup_{\mu\leq\xi} \mathrm{T}_\mu$.
(b) If $a \in \mathrm{T}_\xi$, $b \in \mathrm{T}_\mu$, then $a \prec b :\Leftrightarrow (\xi < \mu \vee (\xi = \mu \wedge a \prec_\xi b))$.
(c) If $a \in \mathrm{T}_\xi$, then $\mathrm{k}(a) := \bigcup_{\mu\in\Sigma} \mathrm{k}_{\xi,\mu}(a)$.
(d) If $A \subseteq \mathrm{T}$, $a \in \mathrm{T}$, $A \cap a := \{b \in A \mid b \prec a\}$

It seems, as if we are back to the traditional way of ordinal notation systems, in which small ordinals (e.g. $a \in \mathrm{T}_0$) are denoted by referring to bigger ones (e.g. $b \in \mathrm{k}(a) \cap \mathrm{T}_1$, therefore $a \prec b$). But the real picture to keep in mind should be that of several ordinal systems, which refer to each other in a controlled way, i.e. smaller ordinal systems refer to bigger ordinal systems restricted to smaller levels only. The global order will be helpful in order to reduce the amount of syntax.
In the well-ordering proof we defined by recursion on $\xi \in \Sigma$ sets $\mathrm{M}_\xi, \mathrm{Acc}_\xi \subseteq \mathrm{T}_\xi$ by:

$$\begin{aligned}\mathrm{M}_\xi &:= \mathrm{T}_\xi \restriction_{\mu<\xi} \mathrm{Acc}_\nu,\\ \mathrm{Acc}_\xi &:= \mathrm{Acc}_\prec(\mathrm{M}_\xi).\end{aligned}$$

We can now use transfinite sequences of ordinal notations in order to obtain the same sets. We proceed as follows: we consider in increasing order all elements of

Σ. For each $\xi \in \Sigma$ we select, as long as possible, successively the least element of T_ξ which has smaller components in the set of ordinals already selected before, but has not been selected itself. Once we cannot find any more such a minimal element, we select the next element of Σ. Once we have finished, the elements of T chosen will be $\bigcup_{\xi \in \Sigma} \mathrm{Acc}_\xi$.
In order to formalize this idea, we first define for sets $A \subseteq \mathrm{T}$ $\mathrm{M}[A] := \{a \in \mathrm{T} \mid \mathrm{k}(a) \cap a \subseteq A\}$ and $\mathrm{M}_\xi[A] := \mathrm{M}[A] \cap \mathrm{T}_\xi$. Now we define by recursion on γ elements $\mathrm{a}_\gamma \in \mathrm{T}$ together with $\mathrm{i}_\gamma \in \Sigma$, $\mathrm{A}_\gamma \subset \mathrm{T}_\gamma$ as follows:

$$\begin{aligned}
\mathrm{A}_\gamma &:= \{\mathrm{a}_\beta \mid \beta < \gamma \wedge \mathrm{a}_\beta \downarrow\}, \\
\mathrm{i}_\gamma &:\simeq \min\nolimits_<\{\xi \in \Sigma \mid \min\nolimits_\prec(\mathrm{M}_\xi[\mathrm{A}_\gamma] \setminus \mathrm{A}_\gamma) \downarrow\}, \\
\mathrm{a}_\gamma &:\simeq \min\nolimits_\prec(\mathrm{M}_{\mathrm{i}_\gamma}[\mathrm{A}_\gamma] \setminus \mathrm{A}_\gamma).
\end{aligned}$$

Convention 2.2. A simultaneous definition of sequences like a_γ, i_γ above should in the following be understood as follows: both i_γ, a_γ (and possibly other sequences) are defined by recursion on the (set theoretic) ordinals γ and can have either value as indicated (above in Σ and T, respectively) or a symbol for undefined. $a \downarrow$ means that a is not a symbol for undefined, $a \uparrow$ means that a is (a symbol for) undefined. Similarly, for a set A $\min_<(A) \downarrow$ ($\min_<(A) \uparrow$) means that a minimum of A with respect to $<$ exists (does not exists). For two possibly objects of a set with an undefined element $a \simeq b :\Leftrightarrow (a \downarrow \vee b \downarrow) \rightarrow a = b$.
In the above definition i_γ is defined iff $\forall \beta < \alpha (\mathrm{i}_\beta \downarrow \wedge \mathrm{a}_\beta \downarrow)$ and a minimum as in the definition above exists. If i_γ is defined, it has as value the corresponding minimal element. The definition of a_γ is similar, but presupposes that i_γ is defined. Similar future definitions are to be understood in the same way.

Therefore we start to define a_γ, i_γ, as long as possible. If this is no longer the case, both get the value undefined. Note that this can be defined in the classical set theory used, in which we can "decide" whether i_γ, a_γ are defined. Because of this definition, this proof is non-constructive.

Lemma 2.3. (Admissibles are Π_2-reflecting)
Assume $\phi_0(x)$, $\phi_1(x,y)$ are Σ-formulas.

(a) *(Provable in Kripke Platek set theory without Δ_0-collection but with natural numbers.) If κ is an admissible ordinal, then from*

$$\exists \alpha < \kappa \,.\, \phi_0(\alpha) \text{ and}$$
$$\forall \alpha < \kappa . \exists \beta < \kappa \,.\, \phi_1(\alpha, \beta)$$

follows the existence of some $\delta < \kappa$ such that

$$\exists \alpha < \delta \,.\, \phi_0(\alpha) \text{ and}$$
$$\forall \alpha < \delta . \exists \beta < \delta \,.\, \phi_1(\alpha, \beta).$$

(b) *In the presence of Δ_0-collection (for the universe of sets) (a) holds for the class of ordinals, i.e.*

$$\exists \alpha \,.\, \phi_0(\alpha) \text{ and}$$

$$\forall\alpha.\exists\beta\ .\ \phi_1(\alpha,\beta)$$

implies the existence of some δ such that the conclusion of (a) holds.
(c) (a), (b) hold as well with ϕ_0, ϕ_1 replaced by several Σ-formulas.

Proof. (a): Define α_n for $n \in \omega$ inductively by:
$\alpha_0 = \alpha + 1$, where α minimal such that $\phi_0(\alpha)$ holds. $n \to n+1$: $\forall\beta < \alpha_n.\exists\gamma < \kappa.\phi_1(\beta,\gamma)$. By Σ-collection there exists $\rho < \kappa$ such that $\forall\beta < \alpha_n.\exists\gamma < \rho.\phi_1(\beta,\gamma)$. Choose $\alpha_{n+1} = \max\{\rho, \alpha_n\}$ for the minimal such ρ. With $\delta := \sup_{n\in\omega}\alpha_n < \alpha$ follows now the assertion.
(b) follows similarly and (c) by coding several Σ-formulas into one.

Definition 2.4. (a) A is T_ξ-*complete* iff

$$(\forall\mu \leq \xi.\forall a \in \mathrm{M}_\mu[A](\mathrm{M}_\mu[A] \cap a \subseteq A) \to a \in A)\ .$$

(b) A is *complete* iff A is T_ξ-complete for all $\xi \in \Sigma$.

Remark 2.5. *(a)* $(\gamma < \gamma' \wedge \mathrm{i}_{\gamma'}\downarrow) \Rightarrow \mathrm{i}_\gamma \leq \mathrm{i}'_\gamma$.
(b) $(\alpha < \beta \wedge \mathrm{i}_\beta\downarrow \wedge \mathrm{i}_\alpha = \mathrm{i}_\beta) \Rightarrow \mathrm{a}_\alpha \prec \mathrm{a}_\beta$
(c) A_β *are $\prec$-well-ordered.*
(d) A_β *is T_ξ-complete iff* $\mathrm{i}_\beta\uparrow$ *or* $\xi \leq \mathrm{i}_\beta$.
A_β *is complete iff* $\mathrm{a}_\beta\uparrow$.
(e) If A_β is T_ξ-complete (complete) and $\beta < \gamma$, then A_γ is T_ξ-complete (complete).

Proof. (a): easy induction on γ'.
(b): Induction on β. $\mathrm{a}_\beta \neq \mathrm{a}_\alpha$. Let $\xi := \mathrm{i}_\alpha$ and assume we had $\mathrm{a}_\beta \prec \mathrm{a}_\alpha$. By (a) $\mathrm{A}_\beta \cap \mathrm{T}_{\prec\xi} = \mathrm{A}_\alpha \cap \mathrm{T}_{\prec\xi}$. By $\mathrm{a}_\beta \prec \mathrm{a}_\alpha$ it follows $\mathrm{a}_\beta \in \mathrm{M}_\xi[\mathrm{A}_\beta] \setminus \mathrm{M}_\xi[\mathrm{A}_\alpha]$, there exists $c \in \mathrm{k}(\mathrm{a}_\beta) \cap \mathrm{T}_\xi$ such that $c \in \mathrm{A}_\beta \setminus \mathrm{A}_\alpha$, $c = \mathrm{a}_\gamma$ for some $\alpha < \gamma < \beta$, $\mathrm{a}_\gamma \prec \mathrm{a}_\beta \prec \mathrm{a}_\alpha$, contradicting the IH.
(c): By (a), (b). (d) Immediate. (e) By (a), (d).

The following Lemma will not be needed for the well-ordering proof. We state it in order to show that we have defined exactly the sets Acc_ξ from the original well-ordering proof.

Lemma 2.6. *(a) If we define* i'_γ, a'_γ, A'_γ *by*

$$\begin{aligned}
\mathrm{A}'_\gamma &:= \{\mathrm{a}'_\beta \mid \beta < \gamma \wedge \mathrm{a}'_\beta\downarrow\},\\
\mathrm{i}'_\gamma &:\simeq \min_<\{\xi \in \Sigma \mid \min_\prec\{a \in \mathrm{T}_\xi \setminus \mathrm{A}'_\gamma \mid \mathrm{k}(a) \cap \mathrm{T}_{<\xi} \subseteq \mathrm{A}'_\gamma\}\downarrow\},\\
\mathrm{a}'_\gamma &:\simeq \min_\prec\{a \in \mathrm{T}_{\mathrm{i}'_\gamma} \setminus \mathrm{A}'_{\mathrm{i}'_\gamma} \mid \mathrm{k}(a) \cap \mathrm{T}_{<\mathrm{i}'_\gamma} \subseteq \mathrm{A}'_\gamma\},
\end{aligned}$$

then $\mathrm{i}'_\gamma \simeq \mathrm{i}_\gamma$, $\mathrm{a}'_\gamma \simeq \mathrm{a}_\gamma$ *for all* γ.
(b) If A_γ is T_ξ-complete, then $\mathrm{A}_\gamma = \mathrm{Acc}_\xi$, where Acc_ξ is defined as in [18].

Proof. (a) Induction on γ. By IH $\mathrm{A}'_\gamma = \mathrm{A}_\gamma$. Let for $\xi \in \Sigma$, $B_\xi := \{a \in \mathrm{T}_\xi \backslash \mathrm{A}_\gamma \mid \mathrm{k}(a) \cap \mathrm{T}_{<\xi} \subseteq \mathrm{A}_\gamma\}$. We show $\min_\prec B_\xi \simeq \min_\prec(\mathrm{M}_\xi[\mathrm{A}_\gamma] \backslash \mathrm{A}_\gamma)$. Assume $c := \min_\prec B_\xi \downarrow$. If $d \in \mathrm{k}(c) \cap \mathrm{T}_\xi$, then $\mathrm{k}(d) \cap \mathrm{T}_{<\xi} \subseteq \mathrm{k}(c) \cap \mathrm{T}_{<\xi} \subseteq \mathrm{A}_\gamma$, by minimality and $d \prec c$ therefore $d \in \mathrm{A}_\gamma$. It follows $c \in \mathrm{M}_\xi[\mathrm{A}_\gamma]$, and by $\mathrm{M}_\xi[\mathrm{A}_\gamma] \subseteq B_\xi$, $c = \min(\mathrm{M}_\xi[\mathrm{A}_\gamma] \setminus \mathrm{A}_\gamma)$. On the other hand, if $c = \min(\mathrm{M}_\xi[\mathrm{A}_\gamma] \setminus \mathrm{A}_\gamma)$, $d \prec c$, $d \in B_\xi$, follows by induction on the length of e $\forall e \in \mathrm{k}(d) \cap \mathrm{T}_\xi . e \in \mathrm{A}_\gamma$ (using $f \in \mathrm{k}(e) \cap \mathrm{T}_\xi \Rightarrow \mathrm{k}(f) \cap \mathrm{T}_{<\xi} \subseteq \mathrm{k}(e)$), therefore $d \in \mathrm{M}_\xi[\mathrm{A}_\gamma] \setminus \mathrm{A}_\gamma$, a contradiction, again $c = \min B_\xi$. Now it follows $\mathrm{i}_\gamma = \mathrm{i}'_\gamma$, $\mathrm{a}_\gamma = \mathrm{a}'_\gamma$.
(b) Induction on ξ. By (a) it follows $\mathrm{A}_\gamma \cap \mathrm{T}_\xi \sqsubseteq \mathrm{M}_\xi$, by Remark 2.5 (c) $\mathrm{A}_\gamma \cap \mathrm{T}_\xi$ is well-ordered, and by completeness and (a) it follows $\mathrm{A}_\gamma = \mathrm{Acc}_\xi$.

Lemma 2.7. *Let τ_ξ be the ξ-th admissible ordinal. If $\alpha \geq \tau_\xi$, then A_α is T_ξ-complete. Especially, the union of all A_α is complete.*

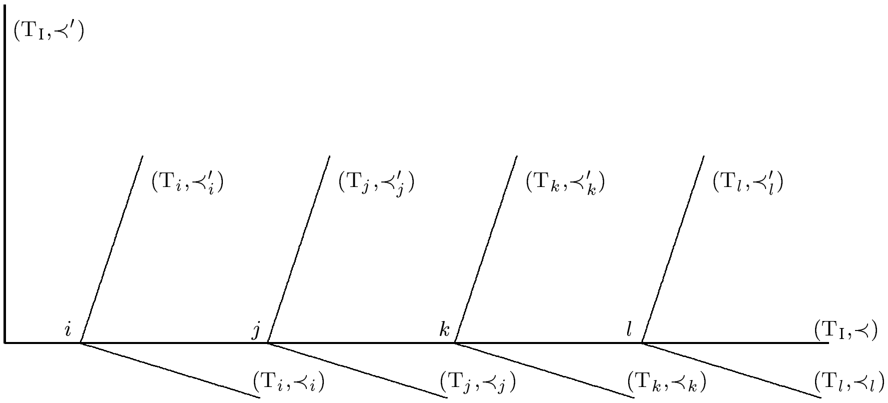
Proof. Induction on ξ. By Remark 2.5 (e) it suffices to show the assertion for $\kappa = \tau_\xi$. By IH A_κ is T_μ-complete ($\mu < \xi$), therefore, if i_κ is defined, then $\xi \leq \mathrm{i}_\kappa$. Assume $c \in \mathrm{M}_\xi[\mathrm{A}_\kappa]$, $\mathrm{M}_\xi[\mathrm{A}_\kappa] \cap c \subseteq \mathrm{A}_\kappa$. Then by the finiteness of $\mathrm{k}(c)$ there exists $\alpha < \kappa$ such that $c \in \mathrm{M}_\xi[\mathrm{A}_\alpha]$ and by Σ-collection relativized to L_κ it follows $\forall \alpha < \kappa . \exists \beta < \kappa (\mathrm{M}_\xi[\mathrm{A}_\alpha] \cap c \subseteq \mathrm{A}_\beta)$. Further $\forall \alpha < \kappa \exists \beta < \kappa . \alpha < \beta$ and $\exists \alpha < \kappa . \alpha = 0$. Application of Lemma 2.3 to the last three formulas, $\exists \alpha < \kappa . c \in \mathrm{M}_\xi[A_\alpha]$ and $\forall \xi' < \xi . \tau_{\xi'} < \kappa$ yields that there exists $\delta < \kappa$ such that the above conditions follow relativized to δ, i.e. δ is a limit ordinal such that $\exists \alpha < \delta . c \in \mathrm{M}_\xi[\mathrm{A}_\alpha]$, $\forall \alpha < \delta . \exists \beta < \delta (\mathrm{M}_\xi[\mathrm{A}_\alpha] \cap c \subseteq \mathrm{A}_\beta)$ and $\forall \xi' < \xi . \tau_{\xi'} < \delta$. It follows $\xi \leq \mathrm{i}_\delta$, $c \in \mathrm{M}_\xi[\mathrm{A}_\delta]$ and $(\mathrm{M}_\xi[\mathrm{A}_\delta] \cap c \subseteq \mathrm{A}_\delta)$, therefore $c \in \mathrm{A}_\delta \subseteq \mathrm{A}_\kappa$ or $c \notin \mathrm{A}_\delta$, $\mathrm{i}_\delta = \xi$, $c = \mathrm{a}_\delta \in \mathrm{A}_\kappa$.

Now we can substitute in the proof of Theorem 4.3 (a) of [18] with $\mathrm{A} := \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{A}_\alpha$ Acc_μ by $\mathrm{A} \cap \mathrm{T}_\mu$ and M_μ by $\mathrm{M}_\mu[\mathrm{A}]$ and replace in the equation which verifies that $\mathrm{C}^\nu(s)$ is definable in ID_σ "$\forall \xi < \mu(\mathrm{k}_{\nu,\xi}(r) \subseteq \mathrm{Acc}_\xi)$" by $\forall \xi \leq \mu(\mathrm{k}_{\nu,\xi}(r) \subseteq \mathrm{Acc}_\xi)$", and show that σ-OS are well-ordered, provable in our variant of Kripke-Platek set theory.

3 Ordinal Systems for one inaccessible

We are going to develop now ordinal systems for one inaccessible, in short I-OS, which will have in the limit order type $|\mathrm{KPI}| = |(\Delta^1_2 - \mathrm{CA}) + (\mathrm{BI})| = \psi(\epsilon_{\mathrm{I}+1})$. The step from σ-OS to I-OS is easy: instead of having one fixed order Σ, this order will now be generated by a further ordinal system.
Therefore we have OS-structures $(\mathrm{T}_a, \prec_a, \prec'_a, \mathrm{k}_{a,a}, \mathrm{length}_a)$ depending on $a \in \mathrm{T}_\mathrm{I}$, where T_I is the underlying set of terms of an OS-structure $(\mathrm{T}_\mathrm{I}, \prec_\mathrm{I}, \prec_\mathrm{I}, \prec'_\mathrm{I}$, $\mathrm{k}_{\mathrm{I},\mathrm{I}}, \mathrm{length}_\mathrm{I})$, functions $\mathrm{k}_{i,j} : \mathrm{T}_i \to_\omega \mathrm{T}_j$, for $i, j \in \mathcal{J} := \mathrm{T}_\mathrm{I} \cup \{\mathrm{I}\}$, an ordered set $(\mathrm{L}, \prec)$ together with functions $\mathrm{level}_i : \mathrm{T}_i \to \mathrm{L}$ and $\widetilde{\mathrm{k}}_i : \mathrm{L} \to_\omega \mathrm{T}_i$ for $i \in \mathcal{J}$. As we did with σ-OS above we will simplify the definitions by replacing the individual component functions $\mathrm{k}_{i,j}$ by only one function $\mathrm{k} : \mathrm{T} \to_\omega \mathrm{T}$, where $\mathrm{T} := \bigcup_{i \in \mathcal{J}} \mathrm{T}_i$, such that for $a \in \mathrm{T}_i$, $\mathrm{k}_{i,j}(a) = \mathrm{k}(a) \cap \mathrm{T}_j$; similarly we define only one function

$\widetilde{\mathrm{k}} : \mathrm{L} \to_\omega \mathrm{T}$, level : $\mathrm{T} \to \mathrm{L}$ and length : $\mathrm{T} \to \omega$, each. Further we will introduce global orders $\prec$, $\prec'$ on T.
The resulting structure can now be visualized as follows:



The conditions are now almost the same as for σ-OS, except for the following two changes:

- If $a \in \mathrm{T}_i$, $i \in \mathrm{T_I}$, then $\mathrm{k}(i) \subseteq \mathrm{k}(a)$. This guarantees that, if from ordinals generated before we can denote a, then we can already denote i. However, we can weaken this condition and will demand it only in case $\mathrm{k}_i(a) \neq \emptyset$ (since otherwise the information about i is already contained in the components of a).
- $\mathrm{k}_j(i) = \emptyset$ for $i \prec j \in \mathrm{T_I}$. This guarantees that in the well-ordering proof using sequences, the sequence i_γ which corresponds to the sequence of i_γ in σ-ordinal systems is increasing, i.e. the elements of $\mathrm{T_I}$ are selected in increasing order.

Definition 3.1. (a) An *Ordinal-System-structure of type* I in short I-OS-*structure* is a 10-tuple

$$(\mathrm{T_I}, (\mathrm{T}_i)_{i\in\mathrm{T_I}}, (\prec_i)_{i\in\mathrm{T_I}\cup\{\mathrm{I}\}}, (\prec'_i)_{i\in\mathrm{T_I}\cup\{\mathrm{I}\}}, \mathrm{length}, \mathrm{k}, \mathrm{L}, \prec_\mathrm{L}, \mathrm{level}, \widetilde{\mathrm{k}})$$

such that, with
- $\mathcal{J} := \mathrm{T_I} \cup \{\mathrm{I}\}$,
- $\mathrm{T} := \bigcup_{i\in\mathcal{J}} \mathrm{T}_i$,

we have
- $\mathrm{T_I}$ and T_i ($i \in \mathrm{T_I}$) are pairwise disjoint sets,
- $\prec_i$, $\prec'_i$ are linear orders on T_i for $i \in \mathcal{J}$,
- $\mathrm{k} : \mathrm{T} \to_\omega \mathrm{T}$,
- length : $\mathrm{T} \to \mathbb{N}$,
- $(\mathrm{L}, \prec_\mathrm{L})$ is a linearly ordered set,
- level : $\mathrm{T} \to \mathrm{L}$,
- $\widetilde{\mathrm{k}} : \mathrm{L} \to_\omega \mathrm{T}$.

(b) For an I-OS-structure as above we define

- $\mathrm{k}_i(a) := \mathrm{k}(a) \cap \mathrm{T}_i$; $\widetilde{\mathrm{k}}_i(l) := \widetilde{\mathrm{k}}(l) \cap \mathrm{T}_i$ $(i \in \mathcal{J},\, a \in \mathrm{T},\, l \in \mathrm{L})$;
- a binary relation $\widetilde{\prec}$ on $\mathcal{J}$ defined for $i, j \in \mathcal{J}$ by

$$i \widetilde{\prec} j :\Leftrightarrow ((i \in \mathrm{T}_\mathrm{I} \wedge j \in \mathrm{T}_\mathrm{I} \wedge i \prec_\mathrm{I} j) \vee (i \in \mathrm{T}_\mathrm{I} \wedge j = \mathrm{I})) \enspace ,$$

- a global order $\prec$ on T defined for $i, j \in \mathcal{J}$, $a \in \mathrm{T}_i$, $b \in \mathrm{T}_j$ by

$$\begin{aligned} a \prec b \Leftrightarrow ((& i = j \wedge a \prec_i b) \vee \\ & (i \in \mathrm{T}_\mathrm{I} \wedge j \in \mathrm{T}_\mathrm{I} \wedge i \prec_\mathrm{I} j) \vee \\ & (i \in \mathrm{T}_\mathrm{I} \wedge j = \mathrm{I} \wedge i \preceq_\mathrm{I} b) \vee \\ & (i = \mathrm{I} \wedge j \in \mathrm{T}_\mathrm{I} \wedge a \prec_\mathrm{I} j)). \end{aligned}$$

- $a \prec' b :\Leftrightarrow a \prec_i b$ for $a, b \in \mathrm{T}_i$.
- If $A \subseteq \mathrm{T}$, then
  $\mathrm{k}^{-1}(A) := \{a \in \mathrm{T} \mid \mathrm{k}(a) \subseteq A\}$,
  $\widetilde{\mathrm{k}}^{-1}(A) := \{a \in \mathrm{L} \mid \widetilde{\mathrm{k}}(a) \subseteq A\}$.
- If $A \subseteq \mathrm{T}$, $a \in \mathrm{T}$, then $A \cap a := \{b \in A \mid b \prec a\}$.

(Note that as for σ-OS $\prec$ should be considered only as a technical simplification: the real intuition should be that several OS-structures are working in parallel; further note that on T_i $(i \in \mathcal{J})$ $\prec$ coincides with $\prec_i$).

(c) An I-OS-structure as above is an I-*times iterated Ordinal System* in short I-OS, if for all $i, j, l \in \mathcal{J}$, $i', j' \in \mathrm{T}_\mathrm{I}$, $r, s \in \mathrm{T}_i$, $r', s' \in \mathrm{T}_{i'}$ the following holds:
(I − OS 1) length$[k(r)] <$ length(r);
(I − OS 2) $\mathrm{k}_i(r) \prec r$;
(I − OS 3) If $r \prec s$, then $r \preceq \mathrm{k}_i(s) \vee r \prec' s \vee$ level$(r) \prec$ level(s).
(I − OS 4) If $j \widetilde{\prec} i$, then $\mathrm{k}_j \circ \mathrm{k}_i \subseteq \mathrm{k}_j$.
(I − OS 5) $\widetilde{\mathrm{k}}(\mathrm{level}(r)) \subseteq \mathrm{k}(r) \cup \widetilde{\mathrm{k}}[\mathrm{level}[\mathrm{k}_i(r)]]$.
(I − OS 6) If $i' \prec j'$, then level$[\mathrm{k}_{j'}(r')] \prec$ level(r').
(I − OS 7) If $r \prec s$, then level$(r) \preceq_\mathrm{L}$ level(s).
(I − OS 8) If $\mathrm{k}_{i'}(r') = \emptyset$, then $\mathrm{k}(i') \subseteq \mathrm{k}(r')$.
(I − OS 9) If $i' \preceq j'$, then $\mathrm{k}_{j'}(i') = \emptyset$.
(I − OS 10) If $A \subseteq \mathrm{T}$ is $\prec$-well-ordered, then $\mathrm{k}^{-1}(A)$ is $\prec'$-well-ordered and $\widetilde{\mathrm{k}}^{-1}(A)$ is $\prec_\mathrm{L}$-well-ordered.

(d) An I-OS as above is *well-ordered* if $(\mathrm{T}, \prec)$ is well-ordered, and its *order type* is that of $(\mathrm{T}_a, \prec_a)$, where a is the $\prec$-minimal element of T_I such that $\mathrm{T}_a \neq \emptyset$, (if no such a exists, the order type is 0).
It is *primitive recursive* if the involved sets are primitive recursive subsets of the natural numbers (parameterized in T_I, i. e. $t \in \mathrm{T}_u$ is primitive recursive in t and u) all functions, relations (including $\prec_\mathrm{I}$) are primitive recursive and all properties (including linearity of $\prec_\mathrm{I}$, $\prec'_\mathrm{I}$) except for (I − OS 10) can be shown in primitive recursive arithmetic.
It is *elementary* if additionally the well-ordering condition follows in PRA in the sense of PRA-reducibility.

Remark 3.2. (I − OS 3), (I − OS 5) *and* (I − *OS* 8) *could be replaced by the following conditions, which will be fulfilled in all the examples given in this article. We have chosen the weaker conditions above in order to be more general.*

(I − OS 3′) *If* $r \prec s$, *then* $r \preceq \mathrm{k}_i(s) \vee r \prec' s$.
(I − OS 5′) $\widetilde{\mathrm{k}}(\mathrm{level}(r)) \subseteq \mathrm{k}(r)$.
(I − OS 8′) $\mathrm{k}(i') \subseteq \mathrm{k}(r')$.

Remark 3.3. *In an* I-*OS as above we have* $\forall a \in \mathrm{T_I}.\mathrm{k}(a) \prec a$.

4 Well-ordering proofs for I-OS using sequences

We are going to show Theorem 6.9 (a), (b): I-OS are well-ordered, and, if they are elementary, they have order type below the proof theoretic strength of KPI. We will show our assertion by giving a well-ordering proof for I-OS, which for elementary I-OS can be formalized in KPI. As mentioned in the introduction, we will carry out these proofs in two versions: In this section we will carry out a well-ordering proof using sequences. We regard this proof as more intuitive than the proof using distinguished sets, which is suitable for constructive theories and carried out in Section 5.
The proof of the parts (a) and (b) of Theorem 6.9 will be almost identical. The latter one is achieved by carrying out the proof of (a) for elementary I-OS in KPI. The only difference between both proofs will be by choosing part (a) or (b) of the following assumption:

Assumption 4.1. In this and the next section assume an I-OS

$$\mathcal{O} = (\mathrm{T_I}, (\mathrm{T}_i)_{i\in\mathcal{J}}, (\prec_i)_{i\in\mathcal{J}}, (\prec'_i)_{i\in\mathcal{J}}, \mathrm{length}, \mathrm{k}, \mathrm{L}, \prec_{\mathrm{L}}, \mathrm{level}, \widetilde{\mathrm{k}})$$

with the usual notations as above. Further assume one of the following:

(a) Our meta-theory is at least Kripke-Platek set theory with natural numbers and there exists an admissible ordinal κ_0 such that $\forall\alpha < \kappa_0 \exists\pi < \kappa_0(\alpha < \pi \wedge \pi$ is an admissible ordinal), where in the following admissible ordinal means "admissible ordinal $> \omega$". $\mathcal{O} \in \mathrm{L}_{\kappa_0}$ and all conditions can be verified in our meta-theory, where condition (I − OS 10) should be provable for all sets $A \in \mathrm{L}_\kappa$. Let in this case Ord := κ_0 and Set := L_{κ_0}, and sets and ordinals range over Ord and Set. This condition is fulfilled if $\mathcal{O}$ is a set and we have as meta-theory ZFC.
(b) Our meta-theory is at least Kripke Platek set theory with natural numbers, $\forall\alpha\exists\kappa.\alpha < \kappa \wedge \kappa$ is an admissible ordinal, $\mathcal{O}$ is a set, all conditions of being an I-OS can be verified in our meta-theory, where condition (I − OS 10) should be provable for all classes A. Let in this case Ord := $\{\alpha \mid \alpha$ ordinal$\}$ and Set := $\{x \mid x = x\}$. This condition holds if we have KPI as meta-theory and $\mathcal{O}$ is elementary.

The well-ordering proof will be similar to the one given in Section 2. Instead of selecting the elements of Σ in increasing order, we will now select for each α the least element of $\mathrm{T_I}$ which has smaller components in the notations already defined before, but was not selected itself. For each such i we first select all elements of T_i as for σ-OS. Once we cannot find any more such an element, we select first i itself and then move on to the next i.

Definition 4.2. (a) $\mathrm{M}[A] := \{a \in \mathrm{T} \mid \mathrm{k}(a) \cap a \subseteq A\}$.
For $i \in \mathcal{J}$ $\mathrm{M}_i[A] := \mathrm{M}[A] \cap \mathrm{T}_i$.

(b) By recursion on the ordinals we define simultaneously $\mathrm{i}_\alpha \in \mathrm{T}_\mathrm{I}$ and $\mathrm{a}_\alpha \in \mathrm{T}_{\mathrm{i}_\alpha} \cup \{\mathrm{i}_\alpha\}$. (Note Convention 2.2.)

$$\begin{aligned} \mathrm{A}_\alpha &:\simeq \{\mathrm{a}_\beta \mid \beta < \alpha \wedge \mathrm{a}_\beta \downarrow\} \\ \mathrm{i}_\alpha &:\simeq \min\nolimits_\prec(\mathrm{M}_\mathrm{I}[\mathrm{A}_\alpha] \setminus \mathrm{A}_\alpha) \\ \mathrm{a}_\alpha &:\simeq \begin{cases} \min_\prec(\mathrm{M}_{\mathrm{i}_\alpha}[\mathrm{A}_\alpha] \setminus \mathrm{A}_\alpha) & \text{if such a minimum exists,} \\ \mathrm{i}_\alpha & \text{otherwise.} \end{cases} \end{aligned}$$

(c) $\mathcal{W} := \{\mathrm{a}_\alpha \mid \alpha \in \mathrm{Ord} \wedge \mathrm{a}_\alpha \downarrow\}$.
$\mathcal{W}_i := \mathcal{W} \cap \mathrm{T}_i$ $(i \in \mathcal{J})$.
$\mathcal{W}_\mathrm{I}^+ := \mathcal{W}_\mathrm{I} \cup \{\mathrm{I}\}$.

Remark 4.3. *If* $A \subseteq \mathrm{M}[A]$, $a \in \mathrm{M}_i[A]$, $i \in \mathrm{T}_\mathrm{I}$, *then* $i \in \mathrm{M}[A]$.

Proof. Induction on length(a). If $\mathrm{k}_i(a) = \emptyset$, then $\mathrm{k}(i) \subseteq \mathrm{k}(a) \cap a$ and therefore the assertion follows. Otherwise $c \in \mathrm{k}_i(a)$ for some c, $c \in A \subseteq \mathrm{M}[A]$, by IH $i \in \mathrm{M}[A]$.

Lemma 4.4. *(a) If* $\alpha < \beta$, $\mathrm{i}_\beta \downarrow$, *then* $\mathrm{i}_\alpha \preceq \mathrm{i}_\beta$.
(b) If $\alpha < \beta$, $\mathrm{a}_\beta \downarrow$, *then* $\mathrm{a}_\alpha \prec \mathrm{a}_\beta$.
(c) A_α *and* $\mathcal{W}$ *are* $\prec$*-well-ordered, i.e. we have the principle of transfinite induction over these sets and classes.*

Proof. (a): Ind on β. Assume β according to the assumption. If $\mathrm{i}_\beta \prec \mathrm{i}_\alpha$, then $\mathrm{i}_\beta \in \mathrm{M}[\mathrm{A}_\beta] \setminus \mathrm{M}[\mathrm{A}_\alpha]$, there exists $c \in \mathrm{k}(\mathrm{i}_\beta) \cap (\mathrm{A}_\beta \setminus \mathrm{A}_\alpha)$. Let $c = \mathrm{a}_\gamma$ with $\alpha \leq \gamma < \beta$. By IH $\mathrm{i}_\beta \prec \mathrm{i}_\alpha \preceq \mathrm{i}_\gamma$, $c = \mathrm{i}_\gamma$ or $c \in \mathrm{T}_{\mathrm{i}_\gamma}$ contradicting $(\mathrm{I} - \mathrm{OS}\ 2)$ and $(\mathrm{I} - \mathrm{OS}\ 9)$.
(b): If $\mathrm{i}_\alpha \prec \mathrm{i}_\beta$, this is clear. Otherwise $\mathrm{i}_\alpha = \mathrm{i}_\beta$, $\mathrm{i}_\gamma = \mathrm{i}_\alpha$ for $\beta \leq \gamma \leq \alpha$ and the assertion follows similarly as in (a).
(c) by (b).

Definition 4.5. (a) $i \in \mathrm{T}_\mathrm{I}$ can be *reached* by A iff $i \in \mathrm{M}_\mathrm{I}[A]$ and $\mathrm{M}_\mathrm{I}[A] \cap i \subseteq A$.

(b) If $i \in \mathrm{T}_\mathrm{I}$, then $a \in \mathrm{T}_i$ can be *reached* by A iff i can be reached by A, $a \in \mathrm{M}_i[A]$ and $\mathrm{M}_i[A] \cap a \subseteq A$ (equivalently: iff $(\mathrm{M}_i[A] \cup \mathrm{M}_\mathrm{I}[A]) \cap a \subseteq A$ and $a, i \in \mathrm{M}_i[A]$).

(c) $A \subseteq \mathrm{T}$ is T_I*-complete* ($\mathrm{T}_\mathrm{I} \cap i$*-complete*) iff, whenever $i \in \mathrm{T}_\mathrm{I}$ ($i \in \mathrm{T}_\mathrm{I} \cap i$) can be reached by A, then $i \in A$.

(d) Assume $i \in \mathrm{T}_\mathrm{I}$. $A \subseteq \mathrm{T}$ is T_i*-complete* iff it is $\mathrm{T}_\mathrm{I} \cap i$-complete and whenever $a \in \mathrm{T}_i$ can be reached by A then $a \in A$.

(e) A is *complete* iff it is T_I-complete and T_i-complete for every $i \in \mathrm{T}_\mathrm{I}$.

Lemma 4.6. *(a)* A_α *is* T_I*-complete iff* $\mathrm{i}_\alpha \uparrow$ *(and therefore as well* $\mathrm{a}_\alpha \uparrow$*).*
(b) If $i \in \mathrm{T}_\mathrm{I}$, *then* A_α *is* $\mathrm{T}_\mathrm{I} \cap i$*-complete iff* $\mathrm{i}_\alpha \uparrow \vee i \preceq \mathrm{i}_\alpha$.
(c) If $i \in \mathrm{T}_\mathrm{I}$, *then* A_α *is* T_i*-complete iff* $\mathrm{i}_\alpha \uparrow \vee i \prec \mathrm{i}_\alpha \vee \mathrm{a}_\alpha = i$.
(d) If A_α *is* T_I*-complete* ($\mathrm{T}_\mathrm{I} \cap i$*-complete,* T_i*-complete, complete) and* $\alpha < \beta$, *then* A_β *is* $\mathrm{T}_\mathrm{I} \cap i$*-complete* ($\mathrm{T}_\mathrm{I} \cap i$*-complete,* T_i*-complete, complete).*

(e) If κ is an admissible ordinal, $\mathcal{O} \in \mathrm{L}_\kappa$, $\mathrm{i}_\kappa \downarrow$, then A_κ is $\mathrm{T}_{\mathrm{i}_\kappa}$-complete and therefore $\mathrm{a}_\kappa = \mathrm{i}_\kappa$.
(f) If $i \in \mathrm{T}_\mathrm{I}$, $a \in \mathcal{W}_i$, then $\mathrm{k}(i) \subseteq \mathcal{W}$.
(g) $\mathcal{W}$ is complete.

Proof. (a), (b), (c): easy. (d): by (a) - (c) and Lemma 4.4 (a).
(e): Let $i := \mathrm{i}_\kappa$. By (b) A_κ is $\mathrm{T}_\mathrm{I} \cap i$-complete. Assume $b \in \mathrm{T}_i$, b can be reached by A_κ. Then

$$i \in \mathrm{M}_\mathrm{I}[\mathrm{A}_\kappa] \wedge \mathrm{M}_\mathrm{I}[\mathrm{A}_\kappa] \cap i \subseteq \mathrm{A}_\kappa,$$
$$b \in \mathrm{M}_i[\mathrm{A}_\kappa] \wedge \mathrm{M}_i[\mathrm{A}_\kappa] \cap b \subseteq \mathrm{A}_\kappa.$$

Similarly as in Lemma 2.7 it follows that for some $\alpha < \kappa$

$$i \in \mathrm{M}_\mathrm{I}[\mathrm{A}_\alpha] \wedge \mathrm{M}_\mathrm{I}[\mathrm{A}_\alpha] \cap i \subseteq \mathrm{A}_\alpha,$$
$$b \in \mathrm{M}_i[\mathrm{A}_\alpha] \wedge \mathrm{M}_i[\mathrm{A}_\alpha] \cap b \subseteq \mathrm{A}_\alpha,$$

$\mathrm{i}_\alpha \preceq i$ therefore $\mathrm{i}_\alpha = i$, $b \in \mathrm{A}_\alpha \subseteq \mathrm{A}_\kappa$ or $b = \mathrm{a}_\alpha \in \mathrm{A}_\kappa$.
(f) Induction on $\mathrm{length}(a)$. If $\mathrm{k}_i(a) = \emptyset$, then $\mathrm{k}(i) \subseteq \mathrm{k}(a) \cap a \subseteq \mathcal{W}$. Otherwise there exists $c \in \mathrm{k}_i(a) \subseteq \mathcal{W}$, and by IH follows the assertion.
(g) Assume $i \in \mathrm{M}_\mathrm{I}[\mathcal{W}] \wedge \mathrm{M}_\mathrm{I}[\mathcal{W}] \cap i \subseteq \mathcal{W}$. As in (e) there exists some α such that

$$i \in \mathrm{M}_\mathrm{I}[\mathrm{A}_\alpha] \wedge \mathrm{M}_\mathrm{I}[\mathrm{A}_\alpha] \cap i \subseteq \mathrm{A}_\alpha \ .$$

It follows A_α is $\mathrm{T}_\mathrm{I} \cap i$-complete. There exists $\alpha < \kappa \in \mathrm{Ord}$, κ an admissible ordinal. By (d) A_κ is $\mathrm{T}_\mathrm{I} \cap \mathrm{i}_\kappa$-complete, $\mathrm{i}_\kappa \uparrow \vee i \preceq \mathrm{i}_\kappa$. If $\mathrm{i}_\kappa \uparrow \vee i \prec \mathrm{i}_\kappa$, it follows $i \in \mathrm{A}_\kappa \subseteq \mathcal{W}$, otherwise $\mathrm{i}_\kappa = i$ and by (e) $i = \mathrm{a}_\kappa \in \mathcal{W}$. If i can be reached by $\mathcal{W}$, then by the argument just given $i = \mathrm{a}_\alpha$ for some α, A_α and therefore as well $\mathcal{W}$ are T_i-complete.

Lemma 4.7. *(a) If $i \in \mathrm{T}_\mathrm{I}$ and $a \in \mathcal{W}_i$, then $i \in \mathrm{M}[\mathcal{W}]$ and $\mathrm{M}[\mathcal{W}] \cap i \subseteq \mathcal{W}$.*
(b) If $a \in \mathcal{W}_i$, then $a \in \mathrm{M}[\mathcal{W}]$ and $\mathrm{M}_i[\mathcal{W}] \cap a \subseteq \mathcal{W}$.

Proof. (a) $a = \mathrm{a}_\gamma$ for some γ. $i = \mathrm{i}_\gamma = \min_\prec(\mathrm{M}_\mathrm{I}[\mathrm{A}_\gamma] \setminus \mathrm{A}_\gamma)$, $i \in \mathrm{M}_\mathrm{I}[\mathrm{A}_\gamma] \subseteq \mathrm{M}_\mathrm{I}[\mathcal{W}]$, and by Lemma 4.4 (a), (I$-$OS 2) and (I$-$OS 9) it follows $\mathrm{M}_\mathrm{I}[\mathcal{W}] \cap i = \mathrm{M}_\mathrm{I}[\mathrm{A}_\gamma] \cap i \subseteq \mathrm{A}_\gamma \subseteq \mathcal{W}$. (b) follows in a similar way.

Definition 4.8. (a) $\widetilde{\mathcal{W}} := \mathrm{k}^{-1}(\mathcal{W})$.
(b) $\widetilde{\mathrm{L}} := \widetilde{\mathrm{k}}^{-1}(\mathcal{W})$.
(c) For $M \subseteq \mathrm{T}$, $M^{\preceq l} := \{a \in M \mid a \in \mathrm{T}_\mathrm{I} \vee \mathrm{level}(a) \preceq l\}$.

Lemma 4.9. $\widetilde{\mathcal{W}} \cap \mathrm{T}_\mathrm{I} \subseteq \mathcal{W}$.

Proof. Note $\widetilde{\mathcal{W}} = \mathrm{M}[\mathcal{W}]$. We show $\forall a \in \widetilde{\mathcal{W}}.a \in \mathcal{W}$ by main induction on $\mathrm{level}(a) \in \widetilde{\mathrm{k}}^{-1}(\mathcal{W})$ (with respect to the well-ordering (!) $\prec_\mathrm{L}$ on it), side-induction on $a \in \widetilde{\mathcal{W}}$ (with respect to the well-ordering (!) $\prec'$ on it) and assume a according to the assertion. $a \in \mathrm{M}[\mathcal{W}]$. We show $\forall c \in \mathrm{M}_\mathrm{I}[\mathcal{W}] \cap a.c \in \mathcal{W}$ and assume $c \in \mathrm{M}_\mathrm{I}[\mathcal{W}] \cap a$. If $\mathrm{level}(c) \prec \mathrm{level}(a)$, by main IH $c \in \mathcal{W}$. Otherwise, if $c \prec' a$, then $\mathrm{level}(c) = \mathrm{level}(a)$, by side-IH $c \in \mathcal{W}$. Otherwise $c \preceq \mathrm{k}(a) \subseteq \mathcal{W}$, $c \in \mathrm{M}_\mathrm{I}[\mathcal{W}] \cap \max(\mathrm{k}(a)) \subseteq \mathcal{W}$. Now a can be reached by $\mathcal{W}$ and by the completeness of $\mathcal{W}$ it follows $a \in \mathcal{W}$.

Proof of Theorem 6.9 (a), (b) using sequences.
First note that $(\widetilde{\mathcal{W}}, \prec)$ and $(\widetilde{\mathrm{L}}, \prec_{\mathrm{L}})$ are well-ordered.
We show by induction on $l \in \widetilde{\mathrm{L}}$

$$\forall l \in \widetilde{\mathrm{L}}.(\mathrm{M}[\mathcal{W}])^{\preceq l} \subseteq \mathcal{W}$$

and assume l according to induction.
We show

$$\forall a \in \widetilde{\mathcal{W}}^{\preceq l}.a \in \mathcal{W}$$

by (side-)induction on $(\widetilde{\mathcal{W}}, \prec')$:
Assume $i \in \mathcal{J}$, $a \in \mathrm{M}_i[\mathcal{W}]$ according to the induction. We define for $j \in \mathcal{J}$, $\mathrm{C}^j(a) \subseteq \mathrm{T}_j$ and $\mathrm{C}(a) \subseteq \mathrm{T}$:

$$\mathrm{C}^j(a) := \begin{cases} \mathcal{W}_j \cap a & \text{if } j \tilde{\prec} i, \\ \mathrm{M}_j[\mathcal{W}] \cap a & \text{if } j = i, \\ \{b \in \mathrm{T}_j \mid (j \neq \mathrm{I} \to \mathrm{level}(b) \prec_{\mathrm{L}} \mathrm{level}(a)) \wedge & \\ \qquad \forall l \tilde{\prec} i (\mathrm{k}_l(b) \subseteq \mathcal{W}_l \cap a) \wedge & \\ \qquad \mathrm{k}_i(b) \subseteq \mathrm{M}_i[\mathcal{W}] \cap a \wedge & \text{if } i \tilde{\prec} j. \\ \qquad \forall l \in \mathcal{J} (i \tilde{\prec} l \tilde{\prec} j \to & \\ \qquad\quad \mathrm{level}[\mathrm{k}_l(b)] \prec_{\mathrm{L}} \mathrm{level}(a))\} & \end{cases}$$

$$\mathrm{C}(a) := \bigcup_{j \in \mathcal{J}} \mathrm{C}^j(a).$$

Note that for $i \tilde{\prec} j$,

$$\mathrm{C}^j(a) = \{b \in \mathrm{T}_j \mid (j \neq \mathrm{I} \to \mathrm{level}(b) \prec_{\mathrm{L}} \mathrm{level}(a)) \wedge \forall l \tilde{\prec} j.\mathrm{k}_l(b) \subseteq \mathrm{C}^l(a)\} \ .$$

We prove

$$\forall j \in \mathcal{J}. i \tilde{\preceq} j \to \mathrm{k}[\mathrm{C}^j(a)] \subseteq \mathrm{C}(a) \ , \tag{1}$$

and assume $i \tilde{\preceq} j$, $b \in \mathrm{C}^j(a)$ and show $\forall c \in \mathrm{k}(b).c \in \mathrm{C}(a)$ by induction on length(c). Assume $c \in \mathrm{T}_l$. If $l \tilde{\prec} j$, then by $b \in \mathrm{C}^j(a)$ it follows $c \in \mathrm{C}^l(a)$. If $l = j = i$, then $c \in \mathcal{W}_i \cap a \subseteq \mathrm{M}_i[\mathcal{W}] \cap a = \mathrm{C}(a)$. Otherwise $i \tilde{\preceq} j \tilde{\preceq} l \neq i$. If $l \neq \mathrm{I}$, level$(c) \preceq_{\mathrm{L}}$ level$(b) \preceq_{\mathrm{L}}$ level(a) and one of the $\preceq_{\mathrm{L}}$ in this chain is actually $\prec_{\mathrm{L}}$, so level$(c) \prec_{\mathrm{L}}$ level(a). In both cases $l \neq \mathrm{I}$ and $l = \mathrm{I}$ it follows for all $k \tilde{\prec} l$ that $\mathrm{k}_k(c) \subseteq \mathrm{k}_k(b)$ and by IH $\mathrm{k}_k(c) \subseteq \mathrm{C}(a)$, $c \in \mathrm{C}^l(a)$.

$$\forall j \in \mathcal{J}. i \tilde{\preceq} j \to \forall b \in \mathrm{C}^j(a).\widetilde{\mathrm{k}}(\mathrm{level}(b)) \subseteq \mathrm{C}(a) \tag{2}$$

follows now by induction on length(b).

$$\mathrm{M}_{\mathrm{I}}[\mathcal{W}] \cap a \subseteq \mathrm{C}(a) \ . \tag{3}$$

If $a \in \mathrm{T}_{\mathrm{I}}$, this follows by definition of $\mathrm{C}^{\mathrm{I}}(a)$. Assume $a \in \mathrm{T}_i$, $i \in \mathrm{T}_{\mathrm{I}}$. We show $\forall c \in \mathrm{M}_{\mathrm{I}}[\mathcal{W}] \cap a.c \in \mathrm{C}(a)$ by induction on length(c). By Remark 3.3 $\mathrm{k}_{\mathrm{I}}(c) \subseteq$

$\mathcal{W} \subseteq \mathrm{M}[\mathcal{W}]$ and by side-side-IH $\mathrm{k_I}(c) \subseteq \mathrm{C}(a)$. Further, if $j \in \mathrm{T_I}$, $d \in \mathrm{k}_j(c)$, then $j \prec c \prec i$, $\mathrm{k}_j(c) \subseteq \mathcal{W}_j \cap a = \mathrm{C}^j(a)$. It follows $c \in \mathrm{C}(a)$.
We show by (side-side-)induction on length(b) simultaneously for all j

$$\forall j \in \mathcal{J}.\forall b \in \mathrm{C}^j(a).(b \in \mathcal{W} \wedge (i \widetilde{\preceq} j \to \mathrm{level}(b) \in \widetilde{\mathrm{L}})) \tag{4}$$

and assume b according to induction, $b \in \mathrm{C}^j(a)$. If $j \widetilde{\prec} i$, this is trivial. Otherwise we have $\mathrm{k}(b) \subseteq \mathrm{C}(a)$ and by IH $\mathrm{k}(b) \subseteq \mathcal{W}$, $b \in \widetilde{\mathcal{W}}$. Further $\widetilde{\mathrm{k}}(\mathrm{level}(b)) \subseteq \mathrm{k}(b) \cup \widetilde{\mathrm{k}}[\mathrm{level}[\mathrm{k}_j(b)]] \subseteq \mathcal{W}$ by IH, $\mathrm{level}(b) \in \widetilde{\mathrm{L}}$. Proof of $b \in \mathcal{W}$:
Case $i = j$: $b \prec a$, therefore $b \preceq \mathrm{k}_i(a)$ or $\mathrm{level}(b) \prec \mathrm{level}(a)$ or $b \prec' a$. If $b \preceq \mathrm{k}_i(a)$, by $\mathrm{k}_i(a) \subseteq \mathcal{W}$ and Lemma 4.7 (b) it follows $b \in \mathcal{W}$. If $\mathrm{level}(b) \prec_{\mathrm{L}} \mathrm{level}(a)$, by main IH and $\mathrm{level}(b) \in \widetilde{\mathrm{L}}$ it follows $b \in \mathcal{W}$. Otherwise $b \prec' a$, $b \in \widetilde{\mathcal{W}}$ and by side-IH $b \in \mathcal{W}$.
Case $i \prec j \neq \mathrm{I}$: $\mathrm{level}(b) \prec \mathrm{level}(a)$ and as before $b \in \mathcal{W}$.
Case $i \prec j = \mathrm{I}$: $b \in \widetilde{\mathcal{W}}$, $b \in \mathcal{W}$.
By (3), (4), $\mathrm{C}^i(a) = \mathrm{M}_i[\mathcal{W}] \cap a$, $a \in \mathrm{M}[\mathcal{W}]$, therefore $i \in \mathrm{T_I} \to i \in \mathrm{M}[\mathcal{W}]$ and by the completeness of $\mathcal{W}$ it follows $a \in \mathcal{W}$.

5 Well-ordering proofs using distinguished sets

In this section we are going to develop the technique for carrying out well-ordering proofs using distinguished sets. This is the technique used for analyzing constructive theories and subsystems of analysis without ordinals. In the case of standard ordinal notation systems of the strength of $|\mathrm{KPI}|$ it has been for instance used by [8] for a well-ordering proof in Feferman's theory T_0 and in [17] for a well-ordering proof for Martin-Löf's Type Theory (which is slightly stronger than $|\mathrm{KPI}|$). It was developed by Buchholz (the first presentation of it can be found in [1]).
A distinguished set is a set A which is built in the same way as the sets A_γ are defined. So $A \cap \mathrm{T_I}$ must be constructed by successively selecting the least element in $\mathrm{M_I}[A]$ not selected before, i.e. A is the set of elements of $\mathrm{Acc}(\mathrm{M_I}[A])$ which are less than some $a \in A$. This can be formalized as $A \cap \mathrm{T_I} = \mathrm{Acc}(\mathrm{M_I}[A])|A$. ($\mathrm{Acc}(M)$ is the accessible part, i.e. largest well-ordered initial segment of the ordered set M). Further, $A \cap \mathrm{T}_i$ is constructed in the same way and we get under the assumption that i can be reached, i.e. $\mathrm{M_I}[A] \cap i \subseteq A$, the condition $A \cap \mathrm{T}_i = \mathrm{Acc}(\mathrm{M}_i[A])|A$. We will need as well versions, where we relativize the set of ordinals needed in the definition of the accessible part and therefore get a Σ-definition of distinguished sets, which can be used in instances of Σ-collection.

Definition 5.1. (a) If $A, B \subseteq \mathrm{T}$, $A|B := \{a \in A \mid \exists b \in B.a \preceq b\}$.
(b) If $A \subseteq \mathrm{T}$, $a \in \mathrm{T}$, then $A|a := A|\{a\}$.
(c) $\mathrm{Acc}_\alpha(A) := \mathrm{A}_\alpha$, where a_γ together with A_γ are defined by recursion on γ as $\mathrm{a}_\gamma :\simeq \min_\prec(A \setminus \mathrm{A}_\gamma)$, $\mathrm{A}_\gamma := \{\mathrm{a}_\beta \mid \beta < \gamma \wedge \mathrm{a}_\beta \downarrow\}$.
(d) $\mathrm{Acc}(A) := \mathrm{Acc}_\kappa(A)$, where κ is the least admissible such that $A \in \mathrm{L}_\kappa$. One easily verifies that $\mathrm{Acc}(A)$ is the largest well-ordered segment of A i.e. the least set M such that $a \in M$ iff $\forall b \in A(b \prec a \to b \in M)$.

(e) $\mathrm{W_I}[A] := \mathrm{Acc}(\mathrm{M_I}[A])$.
If $i \in \mathrm{T_I}$, then $\mathrm{W}_i[A] := \{a \in \mathrm{Acc}(\mathrm{M}_i[A]) \mid \mathrm{M_I}[A] \cap i \subseteq A\}$.
$\mathrm{W}[A] := \bigcup_{i\in\mathcal{J}} \mathrm{W}_i[A]$.
(f) $\mathrm{W}^\alpha_\mathrm{I}[A] := \mathrm{Acc}_\alpha(\mathrm{M_I}[A])$.
If $i \in \mathrm{T_I}$, then $\mathrm{W}^\alpha_i[A] := \{a \in \mathrm{Acc}^\alpha(\mathrm{M}_i[A]) \mid \mathrm{M_I}[A] \cap i \subseteq A\}$.
$\mathrm{W}^\alpha[A] := \bigcup_{i\in\mathcal{J}} \mathrm{W}^\alpha_i[A]$.
(g) $A \subseteq \mathrm{T}$ is *distinguished* (written as $\mathrm{Ag}(A)$, where Ag stands for the original German word "ausgezeichnet" for "distinguished") iff $A = \mathrm{W}[A]|A$.
(h) $\mathrm{Ag}_\alpha(A) :\Leftrightarrow \forall a \in (\mathrm{W}_{\alpha+1}[A] \setminus \mathrm{W}_\alpha[A])(A \prec a) \land A = \mathrm{W}_\alpha[A]|A$.
(i) The class $\mathcal{W}$ is the union of all distinguished sets which are elements of Set. $\mathcal{W}^A$ is the union of all distinguished sets which are elements of A.

Remark 5.2. *(a) If $\forall a \in (\mathrm{W}^{\alpha+1}[A] \setminus \mathrm{W}^\alpha[A]).A \prec a$, then $\mathrm{W}^\alpha[A]|A = \mathrm{W}[A]|A$.*
(b) If A is distinguished, $A \in \mathrm{L}_\kappa$, κ admissible, then there exists $\alpha \prec \kappa$ such that $\forall a \in (\mathrm{W}^{\alpha+1}[A] \setminus \mathrm{W}^\alpha[A]).A \prec a$.
(c) If $A \in \mathrm{L}_\kappa$, κ admissible, then A is distinguished iff $\exists\alpha < \kappa.\mathrm{Ag}_\alpha(A)$ iff $\exists\alpha.\mathrm{Ag}_\alpha(A)$.
(d) If B is an admissible set, $a \in B$, $\forall x \in a.\exists y \in B.(\mathrm{Ag}(y) \land \phi(x,y))$, $\phi(x,y)$ a Σ-formula, then there exists $b \in B$ such that $\forall x \in a.\exists y \in b.(\mathrm{Ag}(y) \land \phi(x,y))$. The same holds in case we have (b) of Assumption 4.1 with B replaced by the class of sets.

Proof. (a) It follows $A \prec a$ for all $a \in \mathrm{W}^\beta[A] \setminus \mathrm{W}^\alpha[A])$ and all $\alpha \prec \beta$.
(b) $\forall a \in A \exists\beta < \kappa.a \in \mathrm{W}^\beta[A]$. By κ admissible and $A \in \mathrm{L}_\kappa$ there exists $\alpha < \kappa$ such that $\forall a \in A.\exists\beta < \alpha.a \in \mathrm{W}^\beta[A]$ and by $A = \mathrm{W}^\kappa[A]|A$, $\forall a \in (\mathrm{W}^{\alpha+1}[A] \setminus \mathrm{W}^\alpha[A]).A \prec a$.
(c): By (a), (b). (d): Replace $\mathrm{Ag}(A)$ by the Σ-formula $\exists\alpha.\mathrm{Ag}_\alpha(A)$.

Lemma 5.3. *If A is distinguished, then A is well-ordered.*

Proof. Let $B := \{i \in \mathrm{T_I} \mid i \in A \lor \mathrm{T}_i \cap A \neq \emptyset\}$.
Assertion: $\forall i \in B.B \cap i \subseteq A$. Proof: Assume $j \prec i \in B$, $j \in B$ and show $j \in A$. If by the definition of $j \in B$ we have $j \in A$, we are done. Otherwise there exists some $c \in \mathrm{T}_j \cap A$. $c \in \mathrm{M}[A]$, $j \in \mathrm{M}[A]$, $i \in \mathrm{W}[A]$ or for some $a \in \mathrm{T}_i$, $a \in \mathrm{W}[A]$, and with $b = i$ or $b = a$, $j \in \mathrm{M_I}[A] \cap b \subseteq A$.
B is well-ordered: Assume $\forall y \in B.(\forall z \in B \cap y.\phi(z)) \to \phi(y)$, $y \in B$. Then $B \cap y = A \cap \mathrm{T_I} \cap y$, which is well-ordered, therefore $\forall z \in B \cap y.\phi(z)$, $\phi(y)$.
Define now $f : A \hookrightarrow \{<i,a> \mid i \in B, a \in (A \cap \mathrm{T}_i) \cup \{\infty\}\} =: C$. $\mathrm{T_I} \ni a \longmapsto <a,\infty>$ and if $i \in \mathrm{T_I}$, $\mathrm{T}_i \ni a \longmapsto <i,a>$. If we take as order on $(A \cap \mathrm{T}_i) \cup \{\infty\}$ the order $(A \cap \mathrm{T}_i, \prec) \otimes \{\infty\}$, then C with the lexicographic order is well-ordered. f is order-preserving, therefore A is well-ordered.

Lemma 5.4. *(a) Assume sets A, B, $\mathrm{Ag}(A)$, $\mathrm{Ag}(B)$, $a \in A$, $a \preceq B$. then $a \in B$.*
(b) If B is a set, $\forall X \in B(X \subseteq \mathrm{T} \land \mathrm{Ag}(X))$, then $\mathrm{Ag}(\bigcup B)$.
(c) A set A is distinguished iff A is an initial segment of $\mathcal{W}$, i.e. $A = \mathcal{W}|A$.
(d) $\mathcal{W}$ is well-ordered.

Proof. (a): We show $\forall a \in A.(a \preceq B \to A|a = B|a)$ by induction on a and assume a according to induction. We show $\forall b \in B(b \prec a \to A|b = B|b)$ by induction on b and assume b according to induction. By main and side-IH it follows $A \cap b = B \cap b$, $b \in \mathrm{W}[B]|b = \mathrm{W}[B \cap b]|b = \mathrm{W}[A \cap b]|b = \mathrm{W}[A]|b = A|b$. Therefore the side-induction is complete, together with main IH it follows $A \cap a = B \cap a$ and the main induction step follows as the side induction step.
(b) If $a \preceq \bigcup B$, $a \preceq C$ for some $C \in B$, by (a) $\bigcup B|a = C|a = \mathrm{W}[C]|a = \mathrm{W}[\bigcup B]|a$, therefore $\bigcup B = \mathrm{W}[\bigcup B]|(\bigcup B)$.
(c) If A is distinguished, then $A \subseteq \mathcal{W}$ and if $a \in \mathcal{W}|A$, $a \in C$ for some distinguished set C, by (a) $C|a = A|a$, $a \in A$. Therefore $\mathcal{W}|A = A$. If $\mathcal{W}|A = A$, then $\forall x \in A.\exists C(\mathrm{Ag}(C) \wedge x \in C)$, By Remark 5.2 (d) there exists D such that $\forall x \in A.\exists C \in D(\mathrm{Ag}(C) \wedge x \in C)$. Let $D := \bigcup\{C|x \mid C \in D \wedge \mathrm{Ag}(C) \wedge x \in C \cap A\}$. Then D is distinguished, $A \subseteq D$, $D \subseteq A$ because A is an initial segment of $\mathcal{W}$, $A = D$.
(d) Assume $\forall x \in \mathcal{W}.(\forall y \in \mathcal{W}(y \prec x \to \phi(y)) \to \phi(x))$, $a \in \mathcal{W}$. Then $a \in A$ for some distinguished set A, A is an initial segment of $\mathcal{W}$, therefore $\forall x \in A.(\forall y \in A(y \prec x \to \phi(y)) \to \phi(x))$, $\forall x \in A.\phi(x)$, $\phi(a)$.

Lemma 5.5. *(a) $\mathcal{W}^A$ is distinguished.*
(b) If A is distinguished, $a \in \mathrm{T}_i$, $i \in \mathrm{T}_\mathrm{I}$, a can be reached by A, then $(A \cap a) \cup \{a\}$ is distinguished.
(c) If A is distinguished, $i \in \mathrm{T}_\mathrm{I}$, i can be reached by A and A is T_i-complete, then $(A \cap i) \cup \{i\}$ is distinguished.

Proof. (a) By Lemma 5.4 (b).
(b) Let $A' := (A \cap a) \cup \{a\}$. We have $\mathrm{W}[A'] \cap a = A' \cap a$: If $b \in \mathrm{W}[A'] \cap a$, in case $b \in \mathrm{T}_i$, $b \in \mathrm{M}_i[A'] \cap a = \mathrm{M}_i[A] \cap a \subseteq A \cap a \subseteq A'$, in case $b \in \mathrm{T}_\mathrm{I}$ again $b \in A'$, and in case $b \in \mathrm{T}_j$, $j \neq i, \mathrm{I}$, then $j \in \mathrm{M}_\mathrm{I}[A] \cap a \subseteq A \cap a$ and by A distinguished and $b \in \mathrm{W}[A] \cap j$, $b \in A \cap a \subseteq A'$. Further $A' \cap a = A \cap a \subseteq \mathrm{W}[A] \cap a = \mathrm{W}[A'] \cap a$. $a \in \mathrm{W}[A']$, $a \in A'$, $\mathrm{W}[A']|A' = \mathrm{W}[A']|a = A'$.
(c) Let $A' := (A \cap i) \cup \{i\}$. As in (b) it follows $\mathrm{W}_\mathrm{I}[A']|A' = A'$, $\mathrm{W}_j[A']|A' = A'$ for $j \in \mathrm{T}_\mathrm{I} \setminus \{i\}$. Further by the T_i-completeness of A $\mathrm{W}_i[A'] = \mathrm{W}_i[A] \subseteq A \cap i \subseteq A'$ therefore $\mathrm{W}_i[A']|A' = A'$, A' is distinguished.

Lemma 5.6. *If A, B are distinguished sets, $A \subseteq B$, $a \in \mathrm{T}$, a can be reached by A, then a can be reached by B as well.*

Proof. Case $a = i \in \mathrm{T}_\mathrm{I}$: Let $C := (\mathrm{T}_\mathrm{I} \cap i) \cup \bigcup_{j \in \mathrm{T}_\mathrm{I} \cap i} \mathrm{T}_J$. We show $A \cap C = B \cap C$: "$\subseteq$" follows by $A \subseteq B$. "$\supseteq$": We show $\forall b \in B \cap C.b \in A$ by induction on b. Subcase $b \in \mathrm{T}_\mathrm{I}$: $b \in \mathrm{M}[B]|b = \mathrm{M}[B \cap b]|b = \mathrm{M}[A \cap b]|b \subseteq \mathrm{M}[A] \cap a \subseteq A$. Subcase $b \in \mathrm{T}_j$, $j \in \mathrm{T}_\mathrm{I} \cap C$: $b \in \mathrm{M}[B]$, therefore $j \in \mathrm{M}[B] \cap C$, and as before $j \in A$, $b \prec j \in A$, $b \in B$ and by Lemma 5.4 (a) it follows $b \in A$. Now $i \in \mathrm{M}[A] \subseteq \mathrm{M}[B]$, and $\mathrm{M}_\mathrm{I}[B] \cap i = \mathrm{M}_\mathrm{I}[B \cap C] \cap i = \mathrm{M}_\mathrm{I}[A \cap C] \cap i \subseteq \mathrm{M}[A] \cap i \subseteq A \subseteq B$.
Case $a \in \mathrm{T}_i$, $i \in \mathrm{T}_\mathrm{I}$: We show $A \cap a = B \cap a$. "$\subseteq$" is clear. "$\supseteq$": We show $\forall b \in B \cap a.b \in A$ by induction on b. If $b \in \mathrm{T}_\mathrm{I}$ or $b \in \mathrm{T}_j$, $j \in \mathrm{T}_j$, $j \neq i$, then the assertion follows by the case $a \in \mathrm{T}_\mathrm{I}$ and since i can be reached by A. If $b \in \mathrm{T}_i$, follows the assertion as in the case $a \in \mathrm{T}_\mathrm{I}$, $b \in \mathrm{T}_\mathrm{I}$. As before follows now the assertion.

Lemma 5.7. *(a) If A is an admissible set, i can be reached by $\mathcal{W}^A$, then $\mathcal{W}^A$ is T_i-complete.*
(b) $\mathcal{W}$ is complete.

Proof. (a) Assume $b \in \mathrm{T}_i$ can be reached by $\mathcal{W}^A$. Show $b \in \mathcal{W}^A$. i and b can be reached, i.e.

$$\begin{gathered} i \in \mathrm{M_I}[\mathcal{W}^A] \wedge \mathrm{M_I}[\mathcal{W}^A] \cap i \subseteq \mathcal{W}^A \\ b \in \mathrm{M}_i[\mathcal{W}^A] \wedge \mathrm{M}_i[\mathcal{W}^A] \cap b \subseteq \mathcal{W}^A \end{gathered}$$

By $\mathrm{k}(i) \cup \mathrm{k}(b)$ finite, closure of distinguished sets under union and the assumption there exists B_0 distinguished, $B_0 \in A$ such that $i, b \in \mathrm{M}[B_0]$. Further we can, using Remark 5.2 (d), inductively define distinguished sets $B_{n+1} \in A$ ($n \in \mathbb{N}$) such that $B_n \subseteq B_{n+1}$, $\mathrm{M_I}[B_n] \cap i \subseteq B_{n+1}$, $\mathrm{M}_i[B_n] \cap b \subseteq B_{n+1}$. Let $B := \bigcup_{n \in \omega} B_n$. B is distinguished, $B \in A$,

$$\begin{gathered} i \in \mathrm{M_I}[B] \wedge \mathrm{M_I}[B] \cap i \subseteq B \\ b \in \mathrm{M}_i[B] \wedge \mathrm{M}_i[B] \cap b \subseteq B \end{gathered}$$

By Lemma 5.5 (b) it follows that $(B \cap b) \cup \{b\}$ is distinguished and $b \in \mathcal{W}_A$.
(b) $\mathcal{W}$ is $\mathrm{T_I}$-complete: Assume i can be reached by $\mathcal{W}$. As in (a) it follows that i can be reached by some distinguished set B. Let A be an admissible set such that $B \in A$. Then $B \subseteq \mathcal{W}_A$, by Lemma 5.6 i can be reached by $\mathcal{W}_A$, by (a) $\mathcal{W}_A$ is T_i-complete and by Lemma 5.5(c) it follows $(\mathcal{W}_A \cap i) \cup \{i\}$ is distinguished, $i \in \mathcal{W}$. If $i \in \mathrm{T_I}$, i can be reached by $\mathcal{W}$, then as in (a) it follows that $\mathcal{W}$ is T_i-complete, and the assertion follows.

Lemma 5.8. *(a) If $i \in \mathrm{T_I}$ and $a \in \mathcal{W}_i$, then $i \in \mathrm{M}[\mathcal{W}]$ and $\mathrm{M}[\mathcal{W}] \cap i \subseteq \mathcal{W}$.*
(b) If $a \in \mathcal{W}_i$, then $a \in \mathrm{M}[\mathcal{W}]$ and $\mathrm{M}_i[\mathcal{W}] \cap a \subseteq \mathcal{W}$.

Proof. Trivial

Proof of Theorem 6.9 (a), (b) using distinguished sets: As before, using $\mathcal{W}$ as defined in this section.

6 I-Ordinal Function Generators

We are going to show that the bound derived before is sharp, i.e. the supremum of the order types of elementary I-OS is |KPI|. As for the systems considered in [18] we will introduce I-ordinal function generators (I-OFGs), which provide a more general framework for defining I-OS. Using these we will define directly a sequence of OS, which could be regarded as "the standard I-OS", in which we use essentially the n-times nested Schütte-Klammersymbole plus Cantor normal form, and which exhaust, although we will not prove that, the strength of |KPI|. We will then interpret directly the system used in [4] as an OFG and this will show the desired lower bound.
Note that the use of ordinals is not really necessary in order to develop the ordinal systems, it only simplifies the description of the ordinal systems slightly, and makes the relationship with the more traditional approach to ordinal notation systems more precise.

Definition 6.1. (a) An I-*ordinal function generator*, in short I-*OFG* is a quadruple

$$\mathcal{O} := (\mathcal{C}, \mathcal{L}, \mathcal{A}, \widetilde{\mathrm{k}}')$$

such that

- $\mathcal{C}$ is a set of ordinals,
- $\mathcal{L} = (\mathrm{L}', \prec_{\mathrm{L}'})$ is a well-ordering,
- $\mathcal{A} = (\mathrm{Arg}_\kappa, \prec'_\kappa, \mathrm{k}'_\kappa, \mathrm{l}'_\kappa, \mathrm{level}'_\kappa)_{\kappa \in \mathcal{C} \cup \{\mathrm{I}\}}$ (I a special symbol),
- $\widetilde{\mathrm{k}}' : \mathrm{L}' \to_\omega \mathrm{Ord}$,
- for $\kappa \in \mathcal{C} \cup \{\mathrm{I}\}$ we have
  - $(\mathrm{Arg}_\kappa, \prec'_\kappa)$ is a well-ordering,
  - $\mathrm{k}'_\kappa, \mathrm{l}'_\kappa : \mathrm{Arg}_\kappa \to_\omega \mathrm{Ord}$,
  - $\mathrm{level}'_\kappa : \mathrm{Arg}_\kappa \to \mathrm{L}'$,
  - $\forall a \in \mathrm{Arg}_\kappa . \mathrm{l}'_\kappa(a) \subseteq \mathrm{k}'_\kappa(a)$,
  - $\forall a, b \in \mathrm{Arg}_\kappa (a \prec'_\kappa b \to \mathrm{level}'_\kappa(a) \preceq_{\mathrm{L}'} \mathrm{level}'_\kappa(b))$,
  - $\forall a \in \mathrm{Arg}_\kappa . \widetilde{\mathrm{k}}'[\mathrm{level}'_\kappa(a)] \subseteq \mathrm{k}'_\kappa(a)$,
- Arg_κ ($\kappa \in \mathcal{C} \cup \{\mathrm{I}\}$) are pairwise disjoint,

We assume in the following that L' and A_κ are disjoint and omit therefore the index κ in k', l', $\prec'$, level' and the index L' in $\prec_{\mathrm{L}'}$. Further let κ, μ, ν be, unless mentioned differently, elements of $\mathcal{I} := \mathcal{C} \cup \{\mathrm{I}\}$, and let $\mathcal{O}$ always be as above.

(b) For $\kappa \in \mathcal{C}$ $\kappa^- := \bigcup_{\pi \in \mathcal{C} \cap \kappa} \pi$ and $\mathrm{O}_\kappa := \kappa \setminus (\kappa^- + 1)$.
$\mathrm{I}^- := 0$, $\mathrm{O}_\mathrm{I} := \mathcal{C}$.
Define the relation $\widetilde{\prec}$ on $\mathcal{I}$ by $\kappa \widetilde{\prec} \mu \Leftrightarrow (\kappa, \mu \in \mathcal{C} \land \kappa < \mu) \lor (\kappa \in \mathcal{C} \land \mu = \mathrm{I})$.
Let Λ be minimal such that $\mathcal{C} < \Lambda$. (In order to avoid the existence of large cardinals, one can assume that $\mathcal{C}$ is a class instead of a set and treat Λ as a symbol for undefined).
$\mathrm{Arg} := \bigcup_{\kappa \in \mathcal{I}} \mathrm{Arg}_\kappa$.

(c) An I-OFG is *cardinality based* iff

- $\mathcal{C}$ is a set of cardinals which has as order type a regular cardinal π such that $\omega < \pi \land \forall \kappa \in \mathcal{C} . \kappa < \pi$ (if we have treated Λ as undefined above, we demand that $\mathcal{C}$ is a proper class),
- for all κ such that κ is not a regular cardinal $> \omega$ we have $\mathrm{Arg}_\kappa = \emptyset$,
- for all $B \subseteq \mathrm{Ord}$ countable ${\mathrm{k}'_\kappa}^{-1}(B)$ is countable,
- for $a \in \mathrm{Arg}_\kappa$, $\mathrm{l}'_\kappa(a) \subseteq \mathrm{k}'_\kappa(a) \cap \mathrm{O}_\kappa$.

In this case we define for $a \in \mathrm{Arg}_\kappa$, $\mathrm{k}'_{\kappa,\nu}(a) := \mathrm{k}'_\kappa(a) \cap \mathrm{O}_\nu$.

(d) If $\mathcal{O}$ as above is a cardinality based I-OFG, we define simultaneously for all $\kappa \in \mathcal{I}$ for $a \in \mathrm{Arg}_\kappa$ by recursion on $\mathrm{level}'_\kappa(a) \in \mathrm{L}'$, side-recursion on $(\mathrm{Arg}_\kappa, \prec'_\kappa)$, $\mathrm{eval}_\kappa(a) \in \mathrm{Ord}$, subsets $\mathrm{C}_\kappa(a), \mathrm{C}^n_\kappa(a) \subseteq \mathrm{Ord}$ ($n \in \omega$) and, whether $\mathrm{NF}_\kappa(a)$ holds, by:
Case $\kappa \in \mathcal{C}$:

- $\mathrm{C}^0_\kappa(a) := (\kappa^- + 1) \cup (\bigcup(\mathrm{k}'_\kappa(a) \cap \mathrm{O}_\kappa)) \cup \mathrm{l}'_\kappa(a)$,
- $\mathrm{C}^{l+1}_\kappa(a) := \mathrm{C}^l_\kappa(a) \cup$

$$\begin{aligned}\{\mathrm{eval}_\nu(b) \mid\ & \nu = \mathrm{I} \lor (\nu = \mathrm{eval}_\mathrm{I}(c) \in \mathrm{C}^l_\kappa(a) \land \mathrm{NF}(c) \land c \in \mathrm{Arg}_\mathrm{I}) \land \\ & \kappa \widetilde{\preceq} \nu \land b \in \mathrm{Arg}_\nu \land \\ & ((\nu = \kappa \land b \prec' a) \lor \mathrm{level}'(b) \prec_{\mathrm{L}'} \mathrm{level}'(a)) \land \\ & \mathrm{k}'(b) \subseteq \mathrm{C}^l_\kappa(a) \land \mathrm{NF}_\nu(b)\},\end{aligned}$$

– $\mathrm{eval}_\kappa(a) := \min(\{\alpha \mid \alpha \notin \mathrm{C}_\kappa(a)\} \cup \{\kappa\})$.

Case $\kappa = \mathrm{I}$:

– $\mathrm{C}^0_{\mathrm{I}}(a) := (\bigcup(\mathrm{k}'_{\mathrm{I}}(a) \cap \mathrm{O}_{\mathrm{I}})) \cup \mathrm{l}'_{\mathrm{I}}(a)$,

– $\mathrm{C}^{l+1}_{\mathrm{I}}(a) := \mathrm{C}^l_{\mathrm{I}}(a) \cup$
$$\bigcup\{\mathrm{eval}_{\mathrm{I}}(b) + 1 \mid b \in \mathrm{Arg}_{\mathrm{I}} \wedge b \prec' a \wedge \wedge \mathrm{k}'(b) \subseteq \mathrm{C}^l_{\mathrm{I}}(a) \wedge \mathrm{NF}_{\mathrm{I}}(b)\},$$

– $\mathrm{eval}_{\mathrm{I}}(a) := \min(\{\alpha \in \mathcal{C} \mid \alpha \notin \mathrm{C}_\kappa(a)\} \cup \{\Lambda\})$.

In both cases

– $\mathrm{C}_\kappa(a) := \bigcup_{n\in\omega} \mathrm{C}^n_\kappa(a)$,

– $\mathrm{NF}_\kappa(a) :\Leftrightarrow \mathrm{k}'(a) \subseteq \mathrm{C}_\kappa(a)$.

(e) $\mathrm{eval}(a) := \mathrm{eval}_\kappa(a)$ for $a \in \mathrm{Arg}_\kappa$, $\kappa \in \mathcal{I}$.
$\mathrm{NF}_\kappa := \{a \in \mathrm{Arg}_\kappa \mid \mathrm{NF}_\kappa(b)\}$, $\mathrm{NF} := \bigcup_{\kappa\in\mathcal{I}} \mathrm{NF}_\kappa$.
For $a, b \in \mathrm{Arg}$ $a \prec b :\Leftrightarrow \mathrm{eval}(a) < \mathrm{eval}(b)$.

(f) Let $\mathcal{O}$ be a cardinality based I-OFG.
We define $\mathrm{T} \subseteq \mathrm{NF}$ inductively by: If $\kappa = \mathrm{I} \vee (c \in \mathrm{T} \cap \mathrm{Arg}_{\mathrm{I}} \wedge \kappa = \mathrm{eval}(c))$, $b \in \mathrm{NF}_\kappa$, $\mathrm{k}'(b) \subseteq \mathrm{eval}[\mathrm{T}]$, then $b \in \mathrm{T}$.
$\mathrm{T}_{\mathrm{I}} := \mathrm{T} \cap \mathrm{Arg}_{\mathrm{I}}$. $\mathrm{T}_a := \mathrm{T} \cap \mathrm{Arg}_{\mathrm{eval}_{\mathrm{I}}(a)}$ for $a \in \mathrm{T}_{\mathrm{I}}$.
$\mathrm{Arg}[\mathrm{T}] := \{a \in \mathrm{Arg} \mid \mathrm{k}'(a) \subseteq \mathrm{eval}[\mathrm{T}]\}$, $\mathrm{Arg}_\kappa[\mathrm{T}] := \mathrm{Arg}[\mathrm{T}] \cap \mathrm{Arg}_\kappa$.
Note $\mathrm{T} \subseteq \mathrm{Arg}[\mathrm{T}]$.
Assuming that $\mathrm{eval} \restriction \mathrm{NF}$ is injective and $n \subseteq \mathrm{eval}(\mathrm{T}_{\mathrm{I}})$, which will be shown below, we define $\mathrm{k}^0, \mathrm{k}^1, \mathrm{k} : \mathrm{Arg}[\mathrm{T}] \to_\omega \mathrm{T}$, $\mathrm{length} : \mathrm{Arg}[\mathrm{T}] \to \mathbb{N}$, $\mathrm{level} : \mathrm{Arg}[\mathrm{T}] \to \mathrm{L}'$ simultaneously by induction on $a \in \mathrm{Arg}[\mathrm{T}]$ as follows: Assume $a \in \mathrm{Arg}_\kappa[\mathrm{T}]$.

– $\kappa \in \mathcal{C} \Rightarrow \mathrm{k}^0(a) := (\mathrm{eval}^{-1}(\mathrm{k}'(a)) \cap \mathrm{T}) \cup \mathrm{k}(\mathrm{eval}^{-1}(\kappa))$.

– $\kappa = \mathrm{I} \Rightarrow \mathrm{k}^0(a) := \mathrm{eval}^{-1}[\mathrm{k}'(a)] \cap \mathrm{T}$.

– $\mathrm{k}^1(a) := \mathrm{k}^0(a) \cup \{d \in \mathrm{k}(c) \mid c \in \mathrm{k}^0(a) \wedge d \prec c\}$.

– $\mathrm{k}(a) := \mathrm{k}^1(a) \cup \mathrm{k}[\mathrm{k}^1(a) \cap \mathrm{Arg}_\kappa]$.

– $\mathrm{length}(a) := \max(\mathrm{length}[\mathrm{k}'(a)] \cup \{-1\}) + 1$.

– $\mathrm{level}(a) := \max(\{\mathrm{level}'(a)\} \cup \mathrm{level}[\mathrm{k}_{\kappa,\kappa}(a)])$.

(g) $\mathcal{J} := \mathrm{T}_{\mathrm{I}} \cup \{\mathrm{I}\}$.
In the following let always $k, l \in \mathcal{J}$.
Let $\prec_k$, $\prec'_k$ be the restriction of $\prec$, $\prec'$ to T_k.
$\mathrm{l}(a) := \mathrm{eval}^{-1}(\mathrm{l}'(a))$.
$\mathrm{k}_k(a) := \mathrm{k}(a) \cap \mathrm{T}_k$, $\mathrm{l}_k(a) := \mathrm{l}(a) \cap \mathrm{T}_k$ $(k \in \mathcal{J})$.
$\mathrm{C}'_{\mathrm{I}}(a) := \mathrm{eval}^{-1}(\mathrm{C}_{\mathrm{I}}(a))$, and for $k \in \mathrm{T}_{\mathrm{I}}$, $\mathrm{C}'_k(a) := \mathrm{eval}^{-1}(\mathrm{C}_{\mathrm{eval}(k)}(a))$.
$\mathrm{L} := \widetilde{\mathrm{k}}^{-1}(\mathrm{eval}[\mathrm{T}])$.
$\prec_{\mathrm{L}}$ be the restriction of $\prec_{\mathrm{L}'}$ to L.
Define $\widetilde{\mathrm{k}} : \mathrm{L} \to_\omega \mathrm{T}$, $\widetilde{\mathrm{k}}(a) := \mathrm{eval}^{-1}(\widetilde{\mathrm{k}}'(a)) \cap \mathrm{T}$.
If $\mathrm{eval}(k)^-$ exists, $k^- := \mathrm{eval}^{-1}(\mathrm{eval}(k)^-)$.

Lemma 6.2. *Let $\mathcal{O}$ be a cardinality based* I-*OFG.*

(a) $\mathrm{C}_\kappa(a)$ *is the least set M such that* $\mathrm{C}^0_\kappa(a) \subseteq M$ *and such that, if* $\nu = \mathrm{I} \vee (\nu = \mathrm{eval}_{\mathrm{I}}(c) \in \mathrm{C}^l_\kappa(a) \wedge \mathrm{NF}(c) \wedge c \in \mathrm{Arg}_{\mathrm{I}})$, $\kappa \widetilde{\preceq} \nu$, $b \in \mathrm{Arg}_\nu$, $\mathrm{NF}_\nu(b)$, $\mathrm{k}'(b) \subseteq M$, $(\nu = \kappa \wedge b \prec' a) \vee \mathrm{level}'(b) \prec \mathrm{level}'(a)$, *then* $\mathrm{eval}_\nu(b) \in \mathrm{M}$.

(b) *If* $\kappa \in \mathcal{C}$, $a \in \mathrm{Arg}_\kappa$, *then* $\mathrm{C}_\kappa(a) \cap \mathrm{O}_\kappa$ *is an initial segment of* O_κ.
Especially $\mathrm{eval}_\kappa(a) = \mathrm{C}_\kappa(a) \cap \kappa$.
If $a \in \mathrm{Arg}_\mathrm{I}$, *then* $\mathrm{C}_\mathrm{I}(a)$ *is an initial segment of* Ord.
(c) *If* $a \in \mathrm{Arg}_\kappa$, *then* $\mathrm{eval}_\kappa(a) \in \mathrm{O}_\kappa$ $(\kappa \in \mathcal{I})$.
(d) *If* $a, b \in \mathrm{NF}_\kappa$, *then* $\mathrm{eval}(a) < \mathrm{eval}(b) \Leftrightarrow (a \prec' b \wedge \mathrm{k}'_{\kappa,\kappa}(a) < \mathrm{eval}(b)) \vee \mathrm{eval}(a) \leq \mathrm{k}'_{\kappa,\kappa}(b)$.
Especially $\mathrm{eval} \restriction \mathrm{NF}_\kappa$ *is injective and therefore* k, k^0, k^1, length *and* level *are well-defined.*
(e) $\mathcal{F} := (\mathrm{T}_\mathrm{I}, (\mathrm{T}_k)_{k \in \mathrm{T}_\mathrm{I}}, (\prec_k)_{k \in \mathcal{J}}, (\prec'_k)_{k \in \mathcal{J}}, \mathrm{length}, \mathrm{k}, \mathrm{L}, \prec_\mathrm{L}, \mathrm{level}, \widetilde{\mathrm{k}})$ *is an* I-*OS-structure.*
(f) *In* $\mathcal{F}$, *the relation* $\prec$ *derived from* $\mathcal{F}$ *being an* I-*OS-structure coincides with the original relation* $\prec$.

Proof. (a) follows, since $\mathrm{k}'(a)$ is finite.
(b): Main induction on $\mathrm{level}(a)$, $\prec'$-side induction on a. Assume the assertion for $\mathrm{level}(b) \prec \mathrm{level}(a)$ or $b \prec' a$. We show $\forall \alpha \in \mathrm{C}^n_\kappa(a) \cap \kappa . \alpha \subseteq \mathrm{C}_\kappa(a)$ by side-side-induction on n: $n = 0$: clear, since $\mathrm{l}'(a) \subseteq \mathrm{k}'_{\kappa,\kappa}(a)$. $n \to n+1$: Case $\kappa = \mathrm{I}$: clear. Otherwise: If $\alpha \in (\mathrm{C}^{n+1}_\kappa(a) \setminus \mathrm{C}^n_\kappa(a)) \cap \mathrm{O}_\kappa$, then $\alpha = \mathrm{eval}_\kappa(b)$ with $\mathrm{k}'(b) \subseteq \mathrm{C}^n_\kappa(a)$ and $b \prec' a$, by side-side-IH and, since $\mathrm{l}'(b) \subseteq \mathrm{k}'(b)$, $\bigcup(\mathrm{k}'(b) \cap \mathrm{O}_\kappa) \cup \mathrm{l}'(b) \subseteq \mathrm{C}_\kappa(a)$, and by an immediate induction $\mathrm{C}^l_\kappa(b) \subseteq \mathrm{C}_\kappa(a)$ for all $l \in \omega$, $\mathrm{C}_\kappa(b) \subseteq \mathrm{C}_\kappa(a)$, therefore $\alpha = \min\{\gamma \mid \gamma \notin \mathrm{C}_\kappa(b)\} \subseteq \mathrm{C}_\kappa(a)$.
(c) $\kappa \neq \mathrm{I}$. By an induction and the regularity of κ it follows that the cardinality of $\mathrm{C}^n_\kappa(a)$ and therefore as well that of $\mathrm{C}_\kappa(a)$ is $< \kappa$. We conclude that there exists an $\alpha < \kappa$ such that $\alpha \notin \mathrm{C}_\kappa(a)$.
$\kappa = \mathrm{I}$: Similarly it follows that there the cardinality of $\mathrm{C}_\mathrm{I}(a)$ is $< \Lambda$. Therefore there exists $\alpha < \Lambda$ such that $\alpha \notin \mathrm{C}_\mathrm{I}(a)$. For some $\alpha < \pi \in \mathcal{C}$ it follows $\pi \notin \mathrm{C}_\mathrm{I}(a) \cap \mathcal{C}$.
(d) "$\Leftarrow$" If $a \prec' b$, $\mathrm{k}'_{\kappa,\kappa}(a) < \mathrm{eval}(b)$, $\mathrm{NF}(a)$, it follows easily $\mathrm{k}'(a) \subseteq \mathrm{C}_\kappa(a) \subseteq \mathrm{C}_\kappa(b)$, $\mathrm{eval}(a) \in \mathrm{C}_\kappa(b) \cap \kappa = \mathrm{eval}(b)$. If $\mathrm{eval}(a) \leq \mathrm{k}'_{\kappa,\kappa}(b)$, the assertion follows by $\mathrm{k}'_{\kappa,\kappa}(b) < \mathrm{eval}(b)$. "$\Rightarrow$" If the right side is false, the right side holds for $b \prec a$ or $a = b$, so the left side is false.
(e) Trivial. (f) Easy.

Convention 6.3. In the following we will use all the definitions introduced for general I-OS-structures for the I-OS-structure of Lemma 6.2 (e).

Lemma 6.4. *Assume* $k \in \mathcal{J}$, $a \in \mathrm{T}_k$.

(a) $b \in \mathrm{k}(a) \Rightarrow \mathrm{k}(b) \cap b \subseteq \mathrm{k}(a)$.
(b) $b \in \mathrm{k}_k(a) \Rightarrow \mathrm{k}(b) \subseteq \mathrm{k}(a)$.
(c) $k = \mathrm{I} \Rightarrow \mathrm{k}(a) \prec a$.
(d) $(k \in \mathrm{T}_\mathrm{I} \wedge b \in \mathrm{C}'_k(a) \setminus (k^- \cup \{k^-\})) \Rightarrow \mathrm{k}(b) \subseteq \mathrm{C}'_k(a)$.
(e) $(k \in \mathcal{J} \wedge a \in \mathrm{T}_k) \Rightarrow \mathrm{k}(a) \subseteq \mathrm{C}'_k(a)$.
(f) *If* $l \in \mathrm{T}_\mathrm{I}$, $k \widetilde{\prec} l$, $b \in \mathrm{T}_l \cap \mathrm{C}'_k(a)$, *then* $\mathrm{level}(b) \prec \mathrm{level}(a)$.
(g) $\mathrm{length}[\mathrm{k}(a)] < \mathrm{length}(a)$.
(h) $\mathrm{k}_k(a) \prec a$.
(i) $(b \in \mathrm{T}_k \wedge a \prec b) \Rightarrow (a \prec' b \vee a \preceq \mathrm{k}_k(b))$.

(j) If $k, l \in \mathcal{J}$, $l \tilde{\prec} k$, *then* $\mathrm{k}_l[\mathrm{k}_k(a)] \subseteq \mathrm{k}(a)$.

(k) $\widetilde{\mathrm{k}}(\mathrm{level}(a)) \subseteq \mathrm{k}(a)$.

(l) $k \tilde{\prec} l \in \mathrm{T_I} \Rightarrow \mathrm{level}[\mathrm{k}_l(a)] \prec \mathrm{level}(a)$.

(m) $(a, b \in \mathrm{T}_k \wedge a \prec b) \Rightarrow \mathrm{level}(a) \preceq \mathrm{level}(b)$.

(n) $(k \in \mathrm{T_I} \wedge a \in \mathrm{T}_k) \Rightarrow \mathrm{k}(k) \subseteq \mathrm{k}(a)$.

(o) $(k, l \in \mathrm{T_I} \wedge k \prec l) \Rightarrow \mathrm{k}_l(k) = \emptyset$.

(p) $\mathcal{F}$ *is an* I-*OS. We call any* I-*OS isomorphic to* $\mathcal{F}$ *an* I-OS based on $\mathcal{O}$.

Proof. (a), (b): Induction on $\mathrm{length}(b)$.
(c): Induction on $\mathrm{length}(a)$. $\mathrm{k}^1(a) \preceq \mathrm{k}^0(a) \subseteq \mathrm{C}'_{\mathrm{I}}(a)$, therefore $\mathrm{k}^1(a) \subseteq \mathrm{C}'_{\mathrm{I}}(a)$. By IH $\mathrm{k}[\mathrm{k}^1(a) \cap \mathrm{Arg}_\kappa] \prec \mathrm{k}^1(a) \prec a$.
(d) Induction on $\mathrm{length}(b)$: $\mathrm{k}^0(b) \subseteq \mathrm{C}'_k(a)$ and by IH it follows the assertion.
(e) In case $k = \mathrm{I}$ this follows by $\mathrm{k}(a) \prec a$ and otherwise it follows similarly as (d), but using (d) instead of the IH.
(f) Induction on $\mathrm{length}(b)$. By (c), (d), $\mathrm{k}_l(b) \subseteq \mathrm{C}_k(a)$ and by IH therefore $\mathrm{level}[\mathrm{k}_l(b)] \prec \mathrm{level}(a)$. Further $\mathrm{level}'(b) \prec \mathrm{level}'(a) \preceq \mathrm{level}(a)$, and therefore $\mathrm{level}(b) \prec \mathrm{level}(a)$.
(g) Trivial. (h) By $\mathrm{k}(a) \subseteq \mathrm{C}'_k(a)$. (i) Lemma 6.2 (d).
(j) If $l \prec k \in \mathrm{T_I}$, $b \in \mathrm{k}_k(a)$, then $\mathrm{k}_l(b) \subseteq \mathrm{k}(b) \cap b \subseteq \mathrm{k}(a)$ by (a). If $l \in \mathrm{T_I}$, $k = \mathrm{I}$, $b \in \mathrm{k}_k(a)$, then by (c) $\mathrm{k}(b) \prec b$, $\mathrm{k}(b) = \mathrm{k}(b) \cap b \subseteq \mathrm{k}(a)$.
(k) Note first that for all a $\widetilde{\mathrm{k}}(\mathrm{level}'(a)) \subseteq \mathrm{k}(a)$. We show by induction on $\mathrm{length}(a)$ the assertion: $\widetilde{\mathrm{k}}(\mathrm{level}(a)) \subseteq \widetilde{\mathrm{k}}(\mathrm{level}'(a)) \cup \widetilde{\mathrm{k}}[\mathrm{level}[\mathrm{k}_k(a)]] \subseteq \mathrm{k}(a) \cup \mathrm{k}[\mathrm{k}_k(a)] \subseteq \mathrm{k}(a)$.
(l): By (c), (e), (f).
(m) Induction on $\mathrm{length}(a) + \mathrm{length}(b)$. If $a \preceq \mathrm{k}_k(b)$, then by IH $\mathrm{level}(a) \preceq \mathrm{level}[\mathrm{k}_k(b)] \preceq \mathrm{level}(b)$. Otherwise $a \prec' b$, $\mathrm{level}'(a) \preceq \mathrm{level}'(b) \preceq \mathrm{level}(b)$. Further by IH $\mathrm{level}[\mathrm{k}_k(a)] \preceq \mathrm{level}(b)$ and therefore $\mathrm{level}(a) \preceq \mathrm{level}(b)$.
(n) Trivial. (o) By (d). (p) By the above.

Definition 6.5. (a) $\Omega_0 := 0$, $\Omega_\alpha := \aleph_\alpha$ for $\alpha \neq 0$.

(b) Let $\mathbf{I}$ be the least strongly inaccessible cardinal. (One can easily replace this by demanding that the class of ordinals has the properties of a strongly inaccessible cardinal).

(c) Define $0_k \in \mathcal{S}_k$ by $0_0 := 0$, $0_{k+1} := \Big(\ \Big)$ (the empty Klammersymbol), where $\mathcal{S}_k$ is as in Example 2.7. (i) of [18].

Example 6.6. (a) Let $l \in \omega$, $l > 0$ be fixed. We define an OFG as follows:
- $\mathcal{C} := \{\Omega_\alpha \mid 0 < \alpha < \mathbf{I}\}$, (where $\Omega_\alpha := \aleph_\alpha$ for $\alpha > 0$),

$$(\mathrm{Arg}_{\Omega_\alpha}, \prec'_{\Omega_\alpha}) := \emptyset \text{ if } \alpha \text{ is not a successor ordinal,}$$

$$(\mathrm{Arg}'_{\Omega_{\alpha+1}}, \prec''_{\Omega_{\alpha+1}}) := \begin{cases} \mathrm{CNF}_{\Omega_1}([0, \Omega_1[,]0, \omega[), & \text{if } \alpha = 0, \\ \mathrm{CNF}_{\Omega_{\alpha+1}}([0, \Omega_{\alpha+1}[,]0, \Omega_\alpha[) \setminus & \\ \quad (\mathrm{CNF}_{\Omega_{\alpha+1}}(\{0\},]0, \Omega_\alpha[) \cup & \\ \quad\quad \{\mathrm{CNF}_{\Omega_{\alpha+1}}(1,1)\}) & \text{otherwise,} \end{cases}$$

$$(\mathrm{Arg}_{\Omega_{\alpha+1}}, \prec'_{\Omega_{\alpha+1}}) := (\mathrm{Arg}'_{\Omega_{\alpha+1}}, \prec''_{\Omega_{\alpha+1}}) \otimes \psi_{\Omega_{\alpha+1}}(\mathcal{S}_l),$$

where $\mathcal{S}_l$ is defined as in Example 2.7. (i) of [18], but restricted to ordinals $< \mathbf{I}$.

- $(\mathrm{Arg}'_{\mathrm{I}}, \prec'_{\mathrm{I}}) := \Omega_{[0,\mathbf{I}[} \otimes \psi_{\mathrm{I}}(\mathcal{S}_l)$.
- $\mathrm{k}'_{\Omega_{\alpha+1}}(a)$, $\mathrm{l}'_{\Omega_{\alpha+1}}$ are defined as k_i, l_i in Example 3.7. (b) of [18], for $\alpha = 0$ as $i = 0$ and for $\alpha > 0$ as $i > 0$,
  $\mathrm{l}'_{\mathrm{I}}(\Omega_1) := \mathrm{k}'_{\mathrm{I}}(\Omega_1) := \emptyset$,
  for $\alpha > 0$, $\mathrm{k}'_{\mathrm{I}}(\Omega_\alpha) := \{\alpha\}$, $\mathrm{l}'_{\mathrm{I}}(\Omega_\alpha) := \emptyset$,
  $\mathrm{k}'_{\mathrm{I}}(\psi_{\mathrm{I}}(A)) := \mathrm{k}^l_0(A)$, where k^l_0 is as in Example 2.7. (i) of [18], $\mathrm{l}'_{\mathrm{I}}(\psi_{\mathrm{I}}(A)) := \emptyset$.
- $(\mathrm{L}', \prec_{\mathrm{L}'}) := \mathcal{S}_l$ (with its order).
- $\mathrm{level}_{\Omega_{\alpha+1}}(\mathrm{CNF}'_{\Omega_{\alpha+1}}(\vec{a})) := 0_l$,
  $\mathrm{level}_{\Omega_{\alpha+1}}(\psi_{\Omega_{\alpha+1}}(A)) := A$,
  $\mathrm{level}_{\mathrm{I}}(\Omega_\alpha) := 0_l$,
  $\mathrm{level}_{\mathrm{I}}(\psi_{\mathrm{I}}(A)) := A$.
- $\widetilde{\mathrm{k}}(A) := \mathrm{k}^l_0(A)$.

Then one easily sees that the above definitions yield a cardinality based I-OFG ($\kappa_0 = \mathbf{I}$). It follows $\mathrm{eval}_{\mathrm{I}}(\Omega_\alpha) = \Omega_\alpha$ ($\alpha > 0$),
$\mathrm{eval}_{\Omega_1}(\mathrm{CNF}_{\Omega_1}(\alpha_1, m_1, \dots, \alpha_n, m_n)) = \omega^{\alpha_1} m_1 + \cdots + \omega^{\alpha_n} m_n$,
$\mathrm{eval}_{\Omega_{\alpha+1}}(\mathrm{CNF}_{\Omega_{\alpha+1}}(\alpha_1, \beta_1, \dots, \alpha_n, \beta_n)) = {\Omega_\alpha}^{\alpha_1}\beta_1 + \cdots + {\Omega_\alpha}^{\alpha_n}\beta_n$.

(b) We can obtain the system of [4] from (a) by making the following changes:
- Let $f_i : \mathcal{S}_i \to \mathrm{Ord}$ be defined as f_i in Example 2.7. (i) of [18], but with basis $\mathbf{I}$.
- Restrict $\psi_{\Omega_{\alpha+1}}(A)$ to A such that $f_l(A) \in \mathrm{C}_{\Omega_{\alpha+1}}(f_l(A))$, where $\mathrm{C}_{\Omega_{\alpha+1}}$ is defined as in [4] (or equivalently such that $\widetilde{\mathrm{k}}(A) \subseteq \mathrm{C}_{\Omega_{\alpha+1}}(f_l(A))$).
- Restrict $\psi_{\mathrm{I}}(A)$ to A such that $f_l(A) \in \mathrm{C}_{\mathbf{I}}(f_l(A))$.
- Insert in $(\mathrm{Arg}_{\Omega_{\alpha+1}}, \prec'_{\Omega_{\alpha+1}})$ between $\mathrm{Arg}'_{\Omega_{\alpha+1}}$ and the "$\psi_{\Omega_{\alpha+1}}$-terms" $\varphi_{]0,\Omega_{\alpha+1}[}([0, \Omega_{\alpha+1}[)$, define $\mathrm{k}'_{\Omega_{\alpha+1}}$, $\mathrm{l}'_{\Omega_{\alpha+1}}$ for these new terms as k, l in Example 2.7. (f) of [18] and $\mathrm{level}(\varphi_a b) := 0_l$. (Note that we have already the ω-function, which corresponds to $\lambda\gamma.\varphi_0\gamma$ in the system of [4]).

Then one can easily see that we obtain again a cardinality based I-OFG ($\kappa_0 = \mathbf{I}$), $\mathrm{eval}_{\Omega_{\alpha+1}}$ on $\mathrm{CNF}_{\Omega_{\alpha+1}}(\vec{\alpha})$, and $\mathrm{eval}_{\mathrm{I}}$ on Ω_α have the same values as in (a), $\mathrm{eval}_{\Omega_{\alpha+1}}(\varphi_\alpha\beta) = \varphi_\alpha\beta$, $\mathrm{eval}_{\Omega_{\alpha+1}}(\psi_{\Omega_{\alpha+1}}(A)) = \psi_{\Omega_{\alpha+1}}(f_l(A))$, $\mathrm{eval}_{\mathrm{I}}(\psi_{\mathrm{I}}(A)) = \psi_{\mathbf{I}}(f_l(A))$, where $\psi_{\Omega_{\alpha+1}}$, $\psi_{\mathbf{I}}$ are as in [4], and the term set of the underlying ordinal system together with its order is isomorphic to the set of terms in [4] which make use of terms $\prec \omega_l(\mathrm{I}+1)$ only (where $\omega_0(a) := a$, $\omega_{l+1}(a) := \omega^{\omega_l(a)}$). Therefore in the limit the order types of these I-OS reach $|\mathrm{KPI}|$.

Remark 6.7. *The (straightforward) generalization of Remark 4.7. in [18] to* I-*OFG holds.*

Lemma 6.8. *(a) For every OFG in the Example 6.6 there exists an elementary OS based on it.*
(b) The supremum of the strength of I-*OS is* $|\mathrm{KPI}|$.

Proof. (a) as for Lemma 2.9. of [18]. (b) The elementary OS based on the sequence of OFG of Example 6.6 (b) (as well those of 6.6 (a)) have in the limit the strength of $|\mathrm{KPI}|$.

Theorem 6.9. *(a) Every* I-*OS is well-ordered.*

(b) *Every elementary* I-*OS has order type below* $|\mathrm{KPI}| = |(\Delta^1_2 - \mathrm{CA}) + (\mathrm{BI})| = \psi(\epsilon_{\mathrm{I}+1})$.
(c) *The bound in (a) is sharp.*

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