

► ANTON SETZER, *Bachmann-Howard from below*.

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We revisit and improve our approach [2] of ordinal systems to generate ordinal notation systems based on elementary operations applied strictly from below. This approach first extends the standard ordinal notation system for Γ_0 , which is based on Cantor Normal Form and the Veblen function, by extending the Veblen function to Schütte Klammer Symbols [1], which are then nested iteratively. We obtain Veblen terms such as

$$\varphi \left(\begin{pmatrix} \alpha_1 & \cdots & \alpha_n \\ \beta_1 & \cdots & \beta_k \\ \gamma_1 & \cdots & \gamma_k \end{pmatrix} \cdots \begin{pmatrix} \delta_1 & \cdots & \delta_l \\ \rho_1 & \cdots & \rho_l \end{pmatrix} \right)$$

The elementary operations consists of the natural numbers with their standard ordering, disjoint union, lexicographic ordering on pairs, lexicographic ordering on strictly descending finite sequences, and forming subsets of orderings. All these operations preserve well-ordering. By iterating elementary operations, we obtain ordinal notation systems of strength up to ϵ_0 .

To go beyond ϵ_0 , we form formal terms defined inductively by applying a combination of elementary operations. This yields two orderings on terms: The first one, denoted $\prec'$, is formed directly from the elementary ordering from the ordering induced by the ordering of the underlying components. The other one, denoted by $\prec$ is defined by

$$s \prec t \Leftrightarrow (s \preceq k(t) \vee (k(s) \prec t \wedge s \prec' t))$$

This construction reaches, in its limit, the Bachmann-Howard Ordinal.

[1] KURT SCHÜTTE, *Kennzeichnung von Ordnungszahlen durch rekursiv erklärte Funktionen*, **Mathematische Annalen**, vol. 127 (1954), no. 1, pp. 15–32.

[2] ANTON SETZER, *Ordinal systems*, **Sets and proofs** (Barry Cooper and John Truss, editors), Cambridge University Press, Cambridge, UK, 1999, pp. 301–331.