An upper bound for the proof theoretical strength of Martin-Löf Type Theory with W-type and one universe

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1 Definition of the formal system of Martin-Löf's type theories

Definition 1.1 *(a) The symbols of Martin-Löf's type theory are infinitely many variables z_i $(i \in \omega)$; the symbols $\Rightarrow$, $:$, $,$, $($, $)$, $=$, $\in$, λ; the term constructors (with their arity in parenthesis) 0 (0), $\underline{r}$ (0), $\underline{n}$ (0), n_k (for each $n < k$, with arity 0), $\underline{n}_k$ (for each $k \in \omega$, with arity 0), S (1), i (1), j (1), p_0 (1), p_1 (1), p (2), sup (2), R (2), Ap (2), $\widetilde{+}$ (2), π (2), σ (2), w (2), D (3), P (3), $\widetilde{i}$ (3), C_n ($n \in \omega$, arity $n+1$); the type constructors with their arity N_k (for each $k \in \omega$, arity 0), N (0), U (0), T (1), $+$ (2), Π (2), Σ (2), W (2) and I (3).*

To make it easier to remember the meaning of the symbols, we give the following hints: $\underline{r}$ is the (unique) element of an identity type I; n_k is the nth element of the finite type N_k with k elements, C the **C***asedistinction for this type; O is the zero, S the* **S***uccessor, P* **P***rimitive recursion or induction over the natural numbers N; i stands for left* **i***nclusion, j for right inclusion, D is the choice in the type $A + B$ of disjoint union of A and B; p_0 and p_1 are the projections, p the pairing for the Σ-type; R the* **R***ecursion) over a W-type; Ap the application of a function (as an element of a Π-type) to an element; $\underline{n}$, $\underline{n}_k$, $\widetilde{i}$, $\widetilde{+}$, σ, π, w are codes for the types N, N_k, I, $+$, Σ, Π, W as elements of the universe U, which become, if the* **T***arski-operator T is applied to them, a type.*

(b) The b-objects are variables, $\lambda x.b$ and $C(b_1, \ldots, b_n)$, if C is a n-ary term or type constructor and b, $b_1, \ldots, b_n$ are b-objects.

The set of free variables of a b-object $FV(b)$ are defined as usual. We write $+$, $\widetilde{+}$ infix (that is $(a + b)$ for $+(a, b)$) $(x)t$ for $\lambda x.t$, $(x, y)t$ for $(x)(y)t$, $(x, y, z)t$ for

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$(x)(y,z)t$, and if $S \in \{\Sigma, \Pi, W, \sigma, \pi, w\}$, $Sx \in s.t := S(s,(x)t)$. Further we write rs for $Ap(r,s)$.
We have the usual conventions about omitting brackets, especially the scope of $\lambda x.$ is as long as possible, for instance $\lambda x.st$ should be read as $\lambda x.(st)$. We define for b-objects $b_1, \ldots, b_n, b$ and variables $x_1, \ldots, x_n$ the simultaneous substitution $b[x_1/b_1, \ldots, x_n/b_n]$ as usual (using the convention, that if $x_i = x_j$ then the first substitution applies) and "$b[x_1/b_1, \ldots, x_n/b_n]$ is an allowed substitution".
α-equality ($=_\alpha$) is defined as usual.

(c) *The set of g-objects (for generalized objects) is inductively defined as: variables x are g-terms; if $n < k$, $n, k \in I\!N$, then n_k is a g-term; and if $k \in I\!N$, then $\underline{n}_k$ is a g-term; if r, s, t are g-terms, $x, y, z, x' \in Var_{ML}$, $x \neq y \neq z \neq x$, then 0, $\underline{r}$, $\underline{n}$, Sr, $\lambda x.r$, $p(r,s)$, $sup(r,s)$, $i(r)$, $j(r)$, $P(r,s,(x,y)t)$, $Ap(r,s)$, $p_0(r)$, $p_1(r)$, $R(r,(x,y,z)s)$, $D(r,(x)s,(x')t)$, $\pi x \in r.s$, $\sigma x \in r.s$, $wx \in r.s$, $r \widetilde{+} s$, $\widetilde{i}(r,s,t)$ are g-terms; if $n \in I\!N$ and r, $s_1, \ldots, s_n$ are g-terms, then $C_n(r, s_1, \ldots, s_n)$ is a g-term.*
Let $Term_{Cl}$ be the set of closed g-terms.

(d) *The g-types are N_k ($k \in \omega$), N, U and $\Pi x \in A.B$, $\Sigma x \in A.B$, $Wx \in A.B$, $A + B$, $I(A,r,s)$, $T(r)$ are g-types. (where A, B are g-types, $x \in Var_{ML}$, r, s g-terms).*

(e) *A g-context-piece is a string $x_1 : A_1, \ldots, x_n : A_n$, where $n \geq 0$, x_i different variables, A_i g-types.*
A g-context is a g-context-piece $x_1 : A_1, \ldots, x_n : A_n$, s.t. $FV(A_i) \subset \{x_1, \ldots, x_{n-1}\}$.
A g-statement is $A = B$ or $s = t : A$ where A, B are g-types, s, t are g-terms.

(f) *R-terms (for Russel-terms) are defined as the g-terms, except, that we replace π, σ, w, $\widetilde{+}$, $\widetilde{i}$ by $\Pi, \Sigma, W, +, I$ respectively.*
R-types are defined by the same definition as the g-types, but referring to r-terms instead of g-terms.
R-context-pieces, -contexts, -statements are defined as the corresponding g-constructions, but referring to r-terms and -types instead of the g-terms and -types.

(g) *We treat the usual statements $A :$ type and $s : A$ as abbreviations: $A :$ type $:\equiv A = A$, $s : A :\equiv s = s : A$. Note, that this is in accordance with the usual formulation, because we have extensionality.*

(h) *We abbreviate $[\vec{x}/\vec{t}] := [x_1/t_1, \ldots, x_n/t_n]$ if $\vec{x} = x_1, \ldots, x_n$ and $\vec{t} = t_1, \ldots, t_n$.*
$[x_1/t_1, \ldots, x_n/t_n] \setminus \{x\} := [\vec{x}/\vec{t}] \setminus \{y\}$ is the result of omitting in $[\vec{x}/\vec{t}]$ the x_i/t_i s.t. $x_i = y$, and $[\vec{x}/\vec{t}] \setminus \{y_1, \ldots, y_m\} := (\cdots(([\vec{x}/\vec{t}] \setminus \{y_1\}) \setminus \{y_2\}) \cdots \setminus \{y_m\})$.

Definition 1.2 *of the extensional Martin-Löf Type Theory with W-type and one Universe.*
We will define the rules, which are of the form

(Rule) $\frac{\Gamma_1 \Rightarrow \Theta_1 \cdots \Gamma_n \Rightarrow \Theta_n}{\Gamma \Rightarrow \Theta}$,

where $\Gamma_1, \ldots, \Gamma_n, \Gamma$ are g-context-pieces, $\Theta_1, \ldots, \Theta_n, \Theta$ are g-judgements ($n = 0$ is allowed) in the version à là Tarski, and r-context-pieces and r-judgements in the version à là Russell.
Then we define for $ML = ML^e_1 W_T$ (the extensional version à là Tarski) or $ML = ML^e_1 W_{T,U}$ (the extensional version à là Tarski with additional rules for the universe, which can be embedded in the version à là Rusell), $ML \vdash \Gamma \Rightarrow \Theta$ inductively by:

If $(Rule)$ *is a rule of* ML *as above,* Δ *is a g-context-piece such that* $\Delta, \Gamma_1, , \ldots, \Delta, \Gamma_n$, Δ, Γ *are g-contexts, and if* $ML \vdash \Delta, \Gamma_i \Rightarrow \Theta_i$ *for* $i = 1, \ldots, n$, *then* $ML \vdash \Delta, \Gamma \Rightarrow \Theta$.
Analogously we define $ML_1^e W_R \vdash \Gamma \Rightarrow \Theta$ *(the extensional version à là Russell) and* $ML_1^e W_{R,U} \vdash \Gamma \Rightarrow \Theta$ $ML_1^e W_{R,U} \vdash \Gamma \Rightarrow \Theta$ *(the version à là Russell corresponding to* $ML_1^e W_{T,U}$*), but referring to r-context-pieces, r-contexts etc.instead of g-context-pieces, g-contexts etc.*
We will write Θ *for* $\Rightarrow \Theta$ *as a premise of a rule.*
In the following, let A, B, A', B' *be g-types (or r-types in the formualtion à là Russell),* $a, b, r, s, t, r_i, s_i, t_i, a', b', r', s', t', r'_i, s'_i, t'_i, t''$ *be g-terms (or r-terms)* θ, θ' *be g-(or r-)judgements* Γ' *be a g-(r-)context-pieces.*
Further let $x, y, z, u \in Var_{ML}$. *Additionally assume for all rules, that all substitution mentioned explicitly are allowed. For instance in the rule* $(N_S^=)$, *assume that* $s_1[x/t, y/P(t, s_0, (x, y)s_1)]$ *and* $A[z/St]$ *are allowed substitutions.*

<u>General rules</u>

(ASS) $\frac{A\ type}{x:A \Rightarrow x:A}$ $\qquad$ $(THIN)$ $\frac{A\ type \qquad \Gamma' \Rightarrow \Theta}{x:A, \Gamma' \Rightarrow \Theta}$

(SYM) $\frac{t=t':A}{t'=t:A}$ $\quad \frac{A=B}{B=A}$ $\qquad$ (SUB) $\frac{x:A, \Gamma' \Rightarrow \Theta \qquad \Rightarrow t:A}{\Gamma'[x/t] \Rightarrow \Theta[x/t]}$

$(TRANS)$ $\frac{t=t':A \qquad t'=t'':A}{t=t'':A}$ $\quad \frac{A=B \qquad B=C}{A=C}$ $\qquad$ $(REPL1)$ $\frac{x:A, \Gamma' \Rightarrow B\ type \qquad \Rightarrow t=t':A}{\Gamma'[x/t] \Rightarrow B[x/t]=B[x/t']}$

$(REPL2)$ $\frac{x:A, \Gamma' \Rightarrow s:B \qquad \Gamma' \Rightarrow t=t':A}{\Gamma'[x/t] \Rightarrow s[x/t]=s[x/t']:B[x/t]}$ $\qquad$ $(REPL3)$ $\frac{t:A \qquad A=B}{t:B}$ $\quad \frac{t=t':A \qquad A=B}{t=t':B}$

$(ALPHA)$ $\frac{x:A, \Gamma' \Rightarrow \theta}{x:A', \Gamma' \Rightarrow \theta}$ $\quad \frac{A\ type}{A=A'}$ $\quad \frac{t:A}{t=t':A}$ $\quad$ *(if* $A =_\alpha A'$, $t =_\alpha t'$*)*

Type introduction rules

(N_k^T) $\quad N_k\ type \quad (k \in I\!N)$ $\qquad$ (N^T) $\quad N\ type$

(Π^T) $\frac{A=A' \qquad x:A \Rightarrow B=B'}{\Pi x \in A.B = \Pi x \in A'.B'}$ $\qquad$ (Σ^T) $\frac{A=A' \qquad x:A \Rightarrow B=B'}{\Sigma x \in A.B = \Sigma x \in A'.B'}$

(W^T) $\frac{A=A' \qquad x:A \Rightarrow B=B'}{Wx \in A.B = Wx \in A'.B'}$ $\qquad$ $(+^T)$ $\frac{A=A' \qquad B=B'}{A+B=A'+B'}$

(I^T) $\frac{A=A' \qquad t=t':A \qquad s=s':A}{I(A,t,s)=I(A',t',s')}$

Introduction rules

(N_k^I) $\quad n_k : N_k \quad (n < k,\ n, k \in I\!N)$ $\qquad$ (N^I) $\quad 0 : N \quad \frac{t=t':N}{St=St':N}$

$(\Pi^I)\quad \dfrac{x{:}A{\Rightarrow}t{=}t'{:}B \quad x{:}A{\Rightarrow}B\ \ type}{\lambda x.t{=}\lambda x.t'{:}\Pi x{\in}A.B}$ $\qquad (\Sigma^I)\quad \dfrac{s{=}s'{:}A \quad t{=}t'{:}B[x/s] \quad x{:}A{\Rightarrow}B\ \ type}{p(s,t){=}p(s',t'){:}\Sigma x{\in}A.B}$

$(W^I)\quad \dfrac{r{=}r'{:}A \quad s{=}s'{:}B[x/r]{\rightarrow}Wx{\in}A.B \quad x{:}A{\Rightarrow}B\ \ type}{sup(r,s){=}sup(r',s'){:}Wx{\in}A.B}$

$(+^I_1)\quad \dfrac{s{=}s'{:}A \quad A\ \ type \quad B\ \ type}{i(s){=}i(s'){:}A{+}B}$ $\qquad (+^I_1)\dfrac{s{=}s'{:}B \quad A\ \ type \quad B\ \ type}{j(s){=}j(s'){:}A{+}B}$

$(I^I)\quad \dfrac{t{=}t'{:}A}{\underline{r}{:}I(A,t,t')}$

Elimination rules

$(N^E_k)\quad \dfrac{t{=}t'{:}N_k \quad s_i{=}s_i'{:}A[x/i_k](i{=}0...k{-}1) \quad x{:}N_k{\Rightarrow}A\ \ type}{C_k(t,s_0,...,s_{k-1}){=}C_k(t',s_0',...,s_{k-1}'){:}A[x/t]} \qquad (k \in I\!N)$

$(N^E)\quad \dfrac{r{=}r'{:}N \quad s{=}s'{:}A[z/0] \quad x{:}N,y{:}A[z/x]{\Rightarrow}t{=}t'{:}A[z/Sx] \quad x{:}N{\Rightarrow}A[z/x]\ \ type}{P(r,s,(x,y)t){=}P(r',s',(x,y)t'){:}A[z/r]}$

$(\Pi^E)\quad \dfrac{r{=}r'{:}A \quad s{=}s'{:}\Pi x{\in}A.B \quad x{:}A{\Rightarrow}B\ \ type}{Ap(s,r){=}Ap(s',r'){:}B[x/r]}$

$(\Sigma^E_0)\quad \dfrac{r{=}r'{:}\Sigma x{\in}A.B \quad x{:}A{\Rightarrow}B\ \ type}{p_0(r){=}p_0(r'){:}A}$ $\qquad (\Sigma^E_1)\quad \dfrac{r{=}r'{:}\Sigma x{\in}A.B \quad x{:}A{\Rightarrow}B\ \ type}{p_1(r){=}p_1(r'){:}B[x/p_0(r)]}$

$(W^E)\quad \dfrac{r{=}r'{:}Wx{\in}A.B \quad x{:}A,y{:}B{\rightarrow}Wx{\in}A.B,z{:}\Pi v{\in}B.C[u/Ap(y,v)]{\Rightarrow}t{=}t'{:}C[u/sup(x,y)] \quad u{:}Wx{\in}A.B{\Rightarrow}C\ \ type}{R(r,(x,y,z)t){=}R(r',(x,y,z)t'){:}C[u/r]}$

$(+^E)\quad \dfrac{r{=}r'{:}A{+}B \quad x{:}A{\Rightarrow}s{=}s'{:}C[z/i(x)] \quad y{:}B{\Rightarrow}t{=}t'{:}C[z/j(y)] \quad z{:}A{+}B{\Rightarrow}C\ \ type}{D(r,(x)s,(y)t){=}D(r',(x)s',(y)t'){:}C[z/r]}$

$(I^E)\quad \dfrac{r{:}I(A,s,t) \quad s{:}A \quad t{:}A}{s{=}t{:}A}$

Equality rules

$(N^=_k)\quad \dfrac{s_i{:}A[x/i_k](i{=}0...k{-}1) \quad x{:}N_k{\Rightarrow}A\ \ type}{C_k(n_k,s_0,...,s_{k-1}){=}s_n{:}A[x/n_k]} \qquad (n < k,\ \ n,k \in I\!N)$

$(N^=_0)\quad \dfrac{s{:}A[z/0] \quad x{:}N,y{:}A[z/x]{\Rightarrow}t{:}A[z/Sx] \quad x{:}N{\Rightarrow}A[z/x]\ \ type}{P(0,s,(x,y)t){=}s{:}A[z/0]}$

$(N^=_S)\quad \dfrac{r{:}N \quad s{:}A[z/0] \quad x{:}N,y{:}A[z/x]{\Rightarrow}t{:}A[z/Sx] \quad x{:}N{\Rightarrow}A[z/x]\ \ type}{P(Sr,s,(x,y)t){=}s_1[x/r,y/P(r,s,(x,y)t)]{:}A[z/Sr]}$

$(\Pi^=)$ $\dfrac{\lambda x.t{:}\Pi x{\in}A.B \quad r{:}A \quad x{:}A{\Rightarrow}B\ \ type}{Ap(\lambda x.t,r){=}t[x/r]{:}B[x/r]}$

(Π^η) $\dfrac{t{:}\Pi x{\in}A.B}{\lambda x.Ap(t,x){=}t{:}\Pi x{\in}A.B}$ $\quad$ *If* $x \notin FV(t)$

$(\Sigma_0^=)$ $\dfrac{p(r,s){:}\Sigma x{\in}A.B \quad A\ \ type}{p_0(p(r,s)){=}r{:}A}$

$(\Sigma_1^=)$ $\dfrac{p(r,s){:}\Sigma x{\in}A.B \quad x{\in}A{\Rightarrow}B\ \ type}{p_1(p(r,s)){=}s{:}B[x/r]}$

$(\Sigma_2^=)$ $\dfrac{t{:}\Sigma x{\in}A.B}{t{=}p(p_0(t),p_1(t)){:}\Sigma x{\in}A.B}$

$(W^=)$ $\dfrac{r{:}A \quad s{:}(B[x/r]{\rightarrow}Wx{\in}A.B) \quad x{:}A,y{:}B{\rightarrow}Wx{\in}A.B,z{:}(\Pi v{\in}B.C[u/Ap(y,v)]){\Rightarrow}t{:}C[u/sup(x,y)] \quad u{:}Wx{\in}A.B{\Rightarrow}C\ \ type x{:}A{\Rightarrow}B\ \ type}{R(sup(r,s),(x,y,z)t){=}t[x/r,y/s,z/\lambda v.R(Ap(s,v),(x,y,z)t)]{:}C[u/sup(r,s)]}$

(*If* $v \notin FV(s) \cup FV((x,y,z)t)$)

$(+_0^=)$ $\dfrac{r{:}A \quad x{:}A{\Rightarrow}s{:}C[z/i(x)] \quad y{:}B{\Rightarrow}t{:}C[z/j(y)] \quad z{\in}A{+}B{\Rightarrow}C\ \ type}{D(i(r),(x)s,(y)t){=}t[x/r]{:}C[z/i(r)]}$

$(+_1^=)$ $\dfrac{r{:}B \quad x{:}A{\Rightarrow}s{:}C[z/i(x)] \quad y{:}B{\Rightarrow}t{:}C[z/j(y)] \quad z{\in}A{+}B{\Rightarrow}C\ \ type}{D(j(r),(x)s,(y)t){=}t[y/r]{:}C[z/j(r)]}$

$(I^=)$ $\dfrac{r{:}I(A,s,t) \quad A\ \ type}{r{=}\underline{r}{:}I(A,s,t)}$

<u>Rules for the universe</u>

Type introduction rules for the universe

(U^I) $\quad U\ \ type$

(T^I) $\dfrac{a{=}a'{:}U}{T(a){=}T(a')}$

Introduction rules for the universe

$(\underline{n}_k^I)$ $\quad \underline{n}_k : U \quad (k \in \omega)$

$(\underline{n}^I)$ $\quad \underline{n} : U$

(π^I) $\dfrac{a{=}a'{:}U \quad x{:}T(a){\Rightarrow}b{=}b'{:}U}{\pi x{\in}a.b{=}\pi x{\in}a'.b'{:}U}$

(σ^I) $\dfrac{a{=}a'{:}U \quad x{:}T(a){\Rightarrow}b{=}b'{:}U}{\sigma x{\in}a.b{=}\sigma x{\in}a'.b'{:}U}$

(w^I) $\dfrac{a{=}a'{:}U \quad x{:}T(a){\Rightarrow}b{=}b'{:}U}{wx{\in}a.b{=}wx{\in}a'.b'{:}U}$

$(\widetilde{+}^I)$ $\dfrac{a{=}a'{:}U \quad b{=}b'{:}U}{a\widetilde{+}b{=}a'\widetilde{+}b'{:}U}$

(i^I) $\dfrac{a{=}a'{:}U \quad s{=}s'{:}T(a) \quad t{=}t'{:}T(a)}{\widetilde{i}(a,s,t){=}\widetilde{i}(a',s',t'){:}U}$

Equality rules for the universe

$(\underline{n}_k^=)$ $\quad T(\underline{n}_k) = N_k \quad (k \in \omega)$

$(\underline{n}^=)$ $\quad T(\underline{n}) = N$

$(\pi^=)$ $\dfrac{a{:}U \quad x{:}T(a){\Rightarrow}b{:}U}{T(\pi x{\in}a.b){=}\Pi x{\in}T(a).T(b)}$

$(\sigma^=)$ $\dfrac{a{:}U \quad x{:}T(a){\Rightarrow}b{:}U}{T(\sigma x{\in}a.b){=}\Sigma x{\in}T(a).T(b)}$

$(w^{=})$ $\frac{a:U \quad x:T(a)\Rightarrow b:U}{T(wx\in a.b)=Wx\in T(a).T(b)}$

$(\widetilde{+}^{=})$ $\frac{a:U \quad b:U}{T(a\widetilde{+}b)=T(a)+T(b)}$

$(i^{=})$ $\frac{a:U \quad t:T(a) \quad s:T(a)}{T(\widetilde{i}(a,t,s))=I(T(a),t,s)}$

The rules of $ML_1^eW_T$ are all rules mentioned above (using g-terms, g-types etc.).
The rules of $ML_1^eW_{T,U}$ are all rules of $ML_1^eW_T$ and additionally the following rules:

(σ^E) $\frac{\sigma x\in a.b:U}{a:U}$ $\frac{\sigma x\in a.b:U}{x:T(a)\Rightarrow b:U}$

(π^E) $\frac{\pi x\in a.b:U}{a:U}$ $\frac{\pi x\in a.b:U}{x:T(a)\Rightarrow b:U}$

(w^E) $\frac{wx\in a.b:U}{a:U}$ $\frac{wx\in a.b:U}{x:T(a)\Rightarrow b:U}$

$(\widetilde{+}^E)$ $\frac{a\widetilde{+}b:U}{a:U}$ $\frac{a\widetilde{+}b:U}{b:U}$

$(i^{=})$ $\frac{i(a,s,t):U}{s:a}$ $\frac{i(a,s,t):U}{t:a}$

The rules of $ML_1^eW_R$ are the same as the rules of $ML_1^eW_T$, but referring to r-terms, r-types etc. and replacing the Rules for the Universe by:

Rules for the universe à là Russell

Type introduction rules for the universe

(U^I) U *type*

$(T^I)\frac{a=a'}{a=a'}$

Introduction rules for the universe

(N_k^U) $N_k : U \quad (k \in \omega)$

(N_k^U) $N : U$

(Π^U) $\frac{a=a':U \quad x:a\Rightarrow b=b':U}{\Pi x\in a.b=\Pi x\in a'.b':U}$

(Σ^U) $\frac{a=a':U \quad x:a\Rightarrow b=b':U}{\Sigma x\in a.b=\Sigma x\in a'.b':U}$

(W^U) $\frac{a=a':U \quad x:a\Rightarrow b=b':U}{Wx\in a.b=Wx\in a'.b':U}$

$(+^U)$ $\frac{a=a':U \quad b=b':U}{a+b=a'+b':U}$

(I^U) $\frac{a=a':U \quad t=t':a \quad s=s':a}{I(a,t,s)=I(a',t',s'):U}$

The rules of $ML_1^eW_{R,U}$ are all rules of $ML_1^eW_R$ and additionally the following rules:

$(\Sigma^{U,E})$ $\frac{\Sigma x\in a.b:U}{a:U}$ $\frac{\Sigma x\in a.b:U}{x:a\Rightarrow b:U}$

$(\Pi^{U,E})$ $\frac{\Pi x\in a.b:U}{a:U}$ $\frac{\Pi x\in a.b:U}{x:a\Rightarrow b:U}$

$(W^{U,E})$ $\frac{Wx\in a.b:U}{a:U}$ $\frac{Wx\in a.b:U}{x:a\Rightarrow b:U}$

$(+^{U,E})$ $\frac{a+b:U}{a:U}$ $\frac{a+b:U}{b:U}$

$(i^{U,E}$ $\frac{I(a,s,t):U}{s:a}$ $\frac{I(a,s,t):U}{t:a}$

2 Comparison of the formulation à là Russell and the formulation à là Tarski

In this section we prove, that $ML_1^eW_{R,U}$ can be embeddded in $ML_1^eW_{T,U}$. Therefore, the upper bound proved for $ML_1^eW_{T,U}$ is an upper bound for $ML_1^eW_R$ and $ML_1^eW_{R,U}$ as well.

Definition 2.1 *(a) Define for C constructors,*

$\psi(N) :\equiv \underline{n}$ $\quad \psi(\Sigma) :\equiv \sigma,$ $\quad \psi(I) :\equiv \widetilde{i}$
$\psi(N_k) :\equiv \underline{n}_k,$ $\quad \psi(W) :\equiv w,$ $\quad \psi(C) :\equiv C$ *otherwise*
$\psi(\Pi) :\equiv \pi,$ $\quad \psi(+) :\equiv \widetilde{+},$

(b) Define ψ : r-term $\rightarrow$ g-term by recursion on the b-objects:
$\psi(x) :\equiv x$ *($x \in Var_{ML}$),*
$\psi(\lambda x.t) :\equiv \lambda x.\psi(t),$
$\psi(C(t_1, \ldots, t_n)) :\equiv \psi(C)(\phi(t_1), \ldots, \phi(t_n)).$

(c) Define the function ρ : r-type $\rightarrow$ g-type by recursion on the g-types:
$\rho(Sx \in r.s) :\equiv Sx \in \rho(r).\rho(s)$ $(S \in \Sigma, \Pi, W),$
$\rho(r + s) :\equiv \rho(r) + \rho(s),$ $\quad \rho(I(r, s, t)) :\equiv I(\rho(r), \psi(s), \psi(t)),$
$\rho(C) :\equiv C$ *for* $C \in \{N, N_k, U\},$ $\quad \rho(t) :\equiv T(\psi(t))$*, otherwise.*

(d) If $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ is a g-context-piece, then $\rho(\Gamma) :\equiv x_1 : \rho(A_1), \ldots, x_n : \rho(A_n)$

(e) If r, s are g-terms, A is a g-type, then
$\rho(r = s : A) :\equiv (\psi(r) = \psi(s) : \rho(A)),$ $\quad \rho(A = B) :\equiv (\rho(A) = \rho(B)).$

(f) Define μ : g-type $\rightarrow$ g-type by recursion on the g-types:
$\mu(T(sx \in r.s)) :\equiv Sx \in \mu(T(r)).\mu(T(s))$ *(where $s = \sigma, \pi, w$ and $S = \Sigma, \Pi, W$ respectively),*
$\mu(T(r\widetilde{+}s)) :\equiv \mu(T(r)) + \mu(T(s)),$ $\quad \mu(\widetilde{i}(r, s, t))) :\equiv I(\mu(T(r)), s, t),$
$\mu(T(\underline{n})) :\equiv N$ $\quad \mu(T(\underline{n}_k)) :\equiv N_k$
$\mu(r + s) :\equiv \mu(r) + \mu(s),$ $\quad \mu(I(r, s, t)) :\equiv I(\mu(r), s, t),$
$\mu(Sx \in r.s) :\equiv Sx \in \mu(r).\mu(s)$ $(S \in \{\Sigma, \Pi, W\})$
$\mu(t) :\equiv t$*, otherwise.*

Lemma 2.2 *Assume r, s, t, s_i b-objects, $x_i \in Var_{ML}$.*

(a) $FV(t) = FV(\psi(t)) = FV(\rho(t)) = FV(\mu(t)).$

(b) If $t[x_1/s_1, \ldots, x_n/s_n]$ allowed, then $\psi(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$, $\mu(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$, $\rho(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$ are allowed.

(c) If t is a r-term, then $\rho(t) = \mu(T(\psi(t)))$.

(d) If t, s_i are b-objects, then $\psi(t[x_1/s_1, \ldots, x_n/s_n]) = \psi(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$.
If t is a g-type, s_i are g-terms, then $\mu(\mu(t)) = \mu(t)$,
$\mu(t[x_1/s_1, \ldots, x_n/s_n]) = \mu(\mu(t)[x_1/s_1, \ldots, x_n/s_n]),$
If t is an r-type, s_i are r-terms, then
$\rho(t[x_1/s_1, \ldots, x_n/s_n]) = \mu(\rho(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]).$

(e) If t is a g -judgement, -statement, -context, -context-piece, then $\rho(t)$ is an r-judgement, -statement, -context, -context-piece.

(f) $r=_\alpha s \to \psi(r)=_\alpha\psi(s),\ \rho(r)=_\alpha\rho(s),\ \mu(r)=_\alpha\mu(s)$.

(g) *If* r *is a g-term and b-term, then* $\psi(r) = r$.
If r *is a g-type and b-type, then* $\rho(r) = r$.

Lemma 2.3 *In all versions of Martin-Löf Type Theory, we have the following useful derived rule:*

$$\frac{\Gamma, x : A, \Gamma' \Rightarrow \theta \quad \Gamma \Rightarrow A = A'}{\Gamma, x : A', \Gamma' \Rightarrow \theta}$$

Proof:
Let y be a fresh variable. Then by $(THIN)$ $\Gamma, y : A', x : A, \Gamma' \Rightarrow \theta$, easily we have $\Gamma, y : A' \Rightarrow y : A$, By (SUB) $\Gamma, y : A', \Gamma'[x/y] \Rightarrow \theta[x/y]$, and by change of the variable the assertion.

Lemma 2.4 *Let* ML_T *be* $ML_1^e W_T$ *or* $ML_1^e W_{T,U}$ *and* Γ, Γ' *be g-context-pieces,* a, b, t *g-terms,* A, B *g-types,* x *a variable. The following applies:*

(a) *If* $ML_T \vdash \Gamma \Rightarrow s : A$, *or* $ML_T \vdash \Gamma \Rightarrow s = t : A$ *then* $ML_T \vdash \Gamma \Rightarrow A$ *type.*

(b) *If* $ML_T \vdash \Gamma \Rightarrow Sx \in A.B$ *type* $(S \in \{\Sigma, \Pi, W\})$*, then* $ML_T \vdash \Gamma \Rightarrow A$, $ML_T \vdash \Gamma, y : A \Rightarrow B[x/y]$ *type, for all* $y \in Var_{ML} \setminus X$ *for some finite set* X.

(c) *If* $ML_T \vdash \Gamma \Rightarrow A + B$ *type then* $ML_T \vdash \Gamma \Rightarrow A$ *type,* $ML_T \vdash \Gamma \Rightarrow B$ *type.*

(d) *If* $ML_T \vdash \Gamma \Rightarrow I(A, b, c)$ *type, then* $ML_T \vdash \Gamma \Rightarrow b : A$, $ML_T \vdash \Gamma \Rightarrow c : A$.

(e) *If* $ML_T \vdash \Gamma \Rightarrow T(b) :$ *type then* $ML_T \vdash \Gamma \Rightarrow b : U$.

(f) *If* $ML_T \vdash \Gamma \Rightarrow A = B$, *then* $ML_T \vdash \Gamma \Rightarrow A$ *type,* $ML_T \vdash \Gamma \Rightarrow B$ *type.*

(g) *If* $ML_T \vdash \Gamma, x : A, \Gamma \Rightarrow \theta$ *then* $ML_T \vdash \Gamma \Rightarrow A$ *type.*

Note that (f) is trivial, but will be needed in the following as an additional premisse for the induction.

Proof: We first change the calculus, treating $A :$ *type* no longer as an abbreviation for $A = A$. (So we have additional rules, with the new statement $A :$ *type*). Further we add all intensional rules, (ass defined in [TD88] or [Set93]) add the rules

$(REPL1a)$ $\frac{x:A,\Gamma',\Gamma''\Rightarrow B\ type \quad \Rightarrow t=t':A}{\Gamma'[x/t],\Gamma''[x/t']\Rightarrow B[x/t']\ type}$ $\qquad (ALPHA1)$ $\frac{A\ type}{A'\ type}$ if $A=_\alpha A'$

$(REFL)$ $\frac{A\ type}{A=A}$ $\qquad \frac{t:A}{A\ type}$

If for this calculus the theorem is provable, then this calculus is equivalent to the original: If we have a proof in the original calculus, then embed it into the calculus, by applying, whenever we need the removed rules the weak inferences. If we have a proof in the new calculus, the result is a proof in the original calculus, since we only added derived rules. Now the proof follows by induction on the length of the (new) derivation. The only difficult case is (SUB), where the difficulty are the the second conclusion in the cases (b): let the conclusion be for instance $\Gamma, \Gamma'[x/t] \Rightarrow (\Sigma x' \in A.B)[x/t]$. By IH $\Gamma, \Gamma', y : A' \Rightarrow B'[x'/y] :$ *type* for $y \notin X$, therefore $\Gamma, \Gamma'[x/t], y : A'[x/t] \Rightarrow B'[x'/y][x/t]$ for $x \neq y$, $x \notin X$. (the substitution is allowed) Then for $y \notin X \cup \{x\}$, if $x = x'$ or $x \notin FV(B)$ follows $(\Sigma x' \in A.B)[x/t] = \Sigma x' \in A[x/t].B$ and we have the assertion, otherwise $x' \notin FV(t)$ and $B[x'/y][x/t] = B[x/t][x'/y]$.
Similarly we argue in $(REPL)$, the other rules are easy.

Lemma 2.5 *(a)* $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow r = s : U$ *, then* $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow T(r) = \mu(T(r))$ *and and* $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow T(s) = \mu(T(s))$

(b) $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow A = B$ *, then* $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow A = \mu(A)$ *and* $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow B = \mu(B)$

Proof:
(a): Induction on the definition of r b-object. If for instance $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow \sigma x \in a.b : U$, then by the additional rules of $ML_1^eW_{T,U}$ follows $\Gamma \Rightarrow a : U$and $\Gamma, x : T(a) \Rightarrow b : U$, by IH therefore $\Gamma \Rightarrow T(a) = \mu(T(a))$, $\Gamma, x : T(a) \Rightarrow T(b) = \mu(T(b))$, by extensional type introduction follows the assertion, similarly for the other terms, for which $\mu(T(t))$ does something.
(b) Similar, using Lemma 2.4 instead of the new rules.

Lemma 2.6 *If* $ML_1^eW_{R,U} \vdash \Gamma \Rightarrow \theta$ *then* $ML_1^eW_{T,U} \vdash \rho(\Gamma) \Rightarrow \rho(\theta)$.
Especially, if $\Gamma \Rightarrow \theta$ *is a statement of both* $ML_1^eW_{R,U}$ *and* $ML_1^eW_{T,U}$*, then we have:*
If $ML_1^eW_{R,U} \vdash \Gamma \Rightarrow \theta$ *then* $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow \theta$.

Proof: Induction on the derivation.
In most rules, the assertion follows by the same rules.
Difficult rules: (SUB), $(REPL)$: Use 2.2 (d), 2.5 and 2.3.
Equality rules and extensional equality rules: use for the substitution part the same argument.
Second and third rule in (U^I): we conclude $T(\psi(A))$, and using 2.5 and an easy argument follows the assertion.
Universe introduction rules (possibly extensional): easy.

Notation 2.7 *In the followins we will write* ML *for* $ML_1^eW_{T,U}$.

3 Definition of Kpi^+

We introduce now the Kripke-Platek set theory KPi^+, in which we will interpret afterwards ML. For more details on it, the reader should refer to [Bar75], [Jäg79], [Jäg83], [JP82], [Jäg86]and [Poh82].

Definition 3.1 *Definition of Kripke-Platek set theory:*

(a) *Let* L_{KP}*be the classical first-order language, with terms being variables, atomic formulas being* $u \in v$, $\neg(u \in v)$, $Ad(u)$, $\neg Ad(u)$. *The set of Variables should be* $Var_{KP} = \{u_i | i \in N\}$ *(a meta-set),* $u_i \neq u_j$ *for* $i \neq j$.
The formulas are built from atomic formulas by $\wedge$, $\vee$, $\forall$, $\exists$. *We define* $\neg A$ *by the deMorgan's laws. The quantifier in* $\forall x.\phi$ *(*$\exists x.\phi$*) is bounded, if* ϕ *of the form* $x \in v \rightarrow B$ *(*$x \in v \wedge B$*) with* $x \neq v$. *A* Δ_0*-formula is a formula with no unbounded quantifier.*
We abbreviate
$A \rightarrow B :\equiv ((\neg A) \vee B)$,
$\forall x \in v.B :\equiv \forall x.x \in v \rightarrow B$,
$\exists x \in v.B :\equiv \exists x.(x \in v \wedge B)$,
$(u = v) :\equiv ((\forall x \in u.x \in v) \wedge (\forall x \in v.x \in u))$,
$u \notin v :\equiv \neg(u \in v)$,
$tran(u) :\equiv \forall x \in u.\forall y \in x.y \in u$,
$infinite(u) :\equiv \exists x \in u.(x = x) \wedge \forall x \in u.\exists y \in u.x \in y$.
$Inacc(x) :\equiv Ad(x) \wedge \forall y \in x.\exists z \in x.y \in z \wedge Ad(z)$.

$Inacc_n(x) :\equiv \exists x_0, \ldots, x_n.Inacc(x_0) \wedge Ad(x_1) \wedge Ad(x_2) \wedge \cdots \wedge Ad(x_n) \wedge x_0 \in x_1 \wedge x_1 \in x_2 \wedge \cdots \wedge x_{n-1} \in x_n \wedge x = x_n$.

ψ a formula, then ψ^u means the replacing of every unbounded quantifier $\forall v$ by $\forall v \in u$ and $\exists v$ by $\exists v \in u$.
Note, that $Inacc(x)$ expresses, that x is an admissible, closed under admissibles, the ordinal of which is an inaccessible, and $Inacc_n(x)$, that x is an admissible, which is at least the nth admissible above an x s.t. $Inacc(x)$.

(b) *Definition of axiom schemes:*

(Ext)	$\forall x.\forall y.\forall z.x = y \rightarrow (x \in z \rightarrow y \in z) \wedge (Ad(x) \rightarrow Ad(y))$
$(Found)$	$\forall \vec{z}.[\forall x.(\forall y \in x.\phi(y, \vec{z}) \rightarrow \phi(x, \vec{z})) \rightarrow \forall x.\phi(x, \vec{z})]$
	(ϕ an arbitrary formula)
$(Pair)$	$\forall x.\forall y.\exists z.x \in z \wedge y \in z.$
$(Union)$	$\forall x.\exists z.\forall y \in x.\forall u \in y.u \in z.$
$(\Delta_0 - sep)$	$\forall \vec{z}.\forall w.\exists y.[\forall x \in y.(x \in w \wedge \phi(x, \vec{z})) \wedge \forall x \in w.\phi(x, \vec{z}) \rightarrow x \in y]$
	(ϕ a Δ_0-formula).
$(\Delta_0 - coll)$	$\forall \vec{z}.\forall w.[\forall x \in w.\exists y.\phi(x, y, \vec{z}) \rightarrow \exists w'.\forall x \in w.\exists y \in w'.\phi(x, y, \vec{z})]$
	(ϕ a Δ_0-formula).
$(Ad.1)$	$\forall x.Ad(x) \rightarrow tran(x) \wedge \exists w \in x.infinite(w).$
$(Ad.2)$	$\forall x.\forall y.Ad(x) \wedge Ad(y) \rightarrow x \in y \vee x = y \vee y \in x.$
$(Ad.3)$	$\forall x.Ad(x) \rightarrow \psi^x,$
	(ψ an instance of (Pair), (Union), (Δ_0-sep), (Δ_0-coll)).
(Lim)	$\forall x.\exists y.Ad(y) \wedge x \in y.$
(inf)	$\exists x.infinite(x).$
$(+_n)$	$\exists x.Inacc_n(x).$

(c) *KPi^+ is the theory (Ext) + (Found) + (Pair) + (Union) + (Δ_0-sep) + (Δ_0-coll) + (inf) + (Ad.1 - 3) + $\{(+_n) | n \in \omega\}$.*
So Kpi^+ is a theory, which guarantees the existence of one recursive inaccessible, and of finitely many admissibles above it.

Definition 3.2 *(a) Ord is the class of ordinals.*

(b) $\alpha(a) := \bigcup(a \cap Ord)$.

(c) $Ad_1 := \bigcap\{x | Ad(x)\}$, $Ad_2 := \bigcap\{x | Ad(x) \wedge Ad_1 \in x\}$, $Ad_I := \bigcap\{x | Inacc(x)\}$, $Ad_{I,n} := \bigcap\{x | Inacc_n(x)\}$. *Note, that Ad_1, Ad_2, Ad_I, $Ad_{I,n}$ can be defined, since there exists b s.t. $Ad(b)$ or $Inacc(b)$ or $Innacc_n(b)$, and therefore we can replace the class by $\{x \,\eta\, b \cup \{b\} | \cdots\}$.*

(d) $\Omega_1 := \alpha(Ad_1)$, $I := \alpha(Ad_I)$, $I_n := \alpha(Ad_{I,n})$.

(e) $ad(u) := \bigcap(\{c \in Ad_I | Ad(c) \wedge u \in c\} \cup Ad_I)$,
$\alpha^+(u) := \alpha(ad(u))$.

Remark 3.3 *In KPi^+ we have*

(a) $Ad(Ad_1)$, $Ad(Ad_2)$, $Inacc(I)$, $Inacc_n(I_n)$.

(b) $u \in Ad_I \rightarrow ad(u) \in Ad_I \wedge Ad(ad(u)) \wedge u \in ad(u)$.

4 Interpretation of terms and types

A type A will be interpreted basically as a set of pairs of closed terms: $< t, s >\in A^*$ should mean, that t and s are equal elements of this types. We will define A^* as the set of terms, which are by an introductory rule elements of this type, and close it under the reduction rule. For instance, if A^* and B^* are already defined, then

$$(A + B)^* := +^*(A^*, B^*)$$

where again

$$+^*(u, v) = Compl(+^{basis}(u, v))$$
$$+^{basis}(u, v) := \{< i(a), i(a')| < a, a' >\in u\} \cup \{< j(b), j(b') > | < b, b' >\in v\}$$
$$Compl(u) := \{< r, s > |\exists r', s'. < r', s' >\in u \wedge r{\rightarrow_{red}}r' \wedge s{\rightarrow_{red}}s'\}$$

Since we only want to interpret finitely many types, namely those types, which occurr in a certain derivation, we interpret dependent types as Σ-functions, the arguments of which are represented by the free variables of the type, in such a way, that $A^*[x_1/t_1, \ldots, x_n/t_n] = (A[x_1/t_1, \ldots, x_n/t_n])^*$.
The Π-type has an introductory rule with a premisse, where dependency occurrs. The intended meaning of the premisse $x : A \Rightarrow t = t' : B$ is

$$\forall r, r'. < r, r' >\in A^* \rightarrow< t[x/r], t'[x/r'] >\in B^*$$

Further we know

$$< r, r' >\in A^* \Rightarrow B^*[x/r] = B'^*[x/r']$$

Since we have to close it under α-conversion we can therefore define $(\Pi x \in A.B)^* := \Pi^*(A^*, (x)B^*)$, where $\Pi^*(u, f) = Compl(\Pi^{basic}(u, f))$ and

$$\Pi^{basis}(u, f) := \{< \lambda y.t, \lambda y'.t' > |\forall < r, r' >\in u. < t[y/r], t'[y'/r'] >\in f(r)\})$$

For technical reason will add the additional condition $f(r) = f(r')$.
In order to make proofs about the terms easy, we will have deterministic reduction-rules. We will allow e.g. $Ap(\lambda x.r, s){\rightarrow_{red}}r[x/s]$ only, if s is in normal form. Further we donot allow any reductions of $\lambda x.r$, so giving reduction rules generally only for closed terms This simple approach is possible, since, from the definition of $(\Pi x \in A.B)^*$, we see, that, if $< \lambda x.t, \lambda x.t >\in (\Pi x \in A.B)^*$, and if $t \rightarrow t'$ in a general sense for open terms, $t[x/r]{\rightarrow_{red}}t'[x/r]$ for closed terms r. But now, if B^* is closed under $\rightarrow_{red}$, we conclude $< \lambda x.t, \lambda x.t' >\in (\Pi x \in A.B)*$.
The interpretation of the W-type, which represents an inductive definition, is done in the usual way: we take some operator on sets and iterate it up to the closure ordinal, which is the next admissible above the components A and B of it. By the introduction rule, we get as the operator F such that

$$\begin{aligned} F_W(u, f)(v) = \quad & Compl(\{< sup(s, \lambda y.t), sup(s', \lambda y'.t') > | < s, s' >\in u \\ & \wedge\forall < r, r' >\in f(s). < t[y/r], t'[y'/r'] \in v\}) \end{aligned}$$

$(Wx \in A.B)^* := F^{\alpha}_W(A^*, (x)B^*)$. We can choose as α any admissible ordinal s.t. A^* and $(x)B^*$ are elements of L_α. We will take as α I_n, the nth admissible above I, and n is the maximum of $level(A)$ and $level(B)$, which is the number of nestings of W-types in A and

B. Although this ordinal is usually too big, it suffices for our construction.
In the introduction rules for the elements of the universe, e.g.

$$(\pi^I) \qquad \frac{a \in U \quad x \in T(a) \Rightarrow b \in U}{\pi x \in a.b : U}$$

we introduce simultaneously the elements a of the universe and its interpretation $T(a)$ as a type. We will therefore first define a set $\widehat{U}$ of tripels $< a, A, b >$, where a and b are terms, considered as equal elements of the universe, such that $T(a)^* = T(b)^* = A$. Therefore

$$U^* = \{< a, b > | \exists A. < a, A, b > \in \widehat{U}\}$$

and $T(a)^* = f(a)$, where

$$f = \{< a, A > | \exists b. < a, A, b > \in \widehat{U}\}$$

$\widehat{U}$ is again the fixed point of an operator $\widetilde{U}$, so $\widehat{U} = \widetilde{U}^\alpha$ for some admissible α. Since U is closed under the W-type, in the definition of $\widetilde{U}$ we need to go to the next admissible, α must be closed under the step to the next admissible, I is a recursive inaccessible, an admissible closed under admissibles. Here we see, why we need the theory KPi^+: We need, one admissible, closed under admissibles, and ω many admissibles above it.
We want to interpret an intuitionistic theory in a classical one, using some realization. Now if we have a realization interpretation as indices for recursive functions and there is naturally a very easy realization of $\neg\forall x.\exists y.\phi(x, y)$, if $\forall x.\exists y.\phi(x, y)$ is arithmetical formula, such that the y is not recursive in x, then we have an obvious realization: every term $\lambda x.t$ does it — there is no realization of $\forall x.\exists y.\phi(x, y)$. Therefore there are false recursive realizations. But proof carried out in an intuitionistic theory like Martin-Löf Type Theory should not prove false statements. The reason, why it oes it not prove any, is, that we can apply the realizing term$\lambda u.s$ of $\neg\forall x.\exists y.\phi(x, y)$, in some sense to non intuitionistic proofs as well. We can add a constructor, which has a non recursive reduction rule, and gives us the y depending on the x, then we have a realizing term t for $\forall x.\exists y.\phi(x, y)$, and can apply any proof $\lambda u.s$ of $\neg\forall x.\exists y.\phi(x, y)$ to it to get an element of type $\perp$ which is empty. Therefore, adding non recursive constructors, we can achieve, that there is no realizing term for a false formula.
We want to extend our result even for Π^1_1-sentences. Here again we have the problem, that the powerset of the natural numbers, $N \to U$, does not represent all sets in KPi^+. We will not be able to get a result, where we conclude from $ML\forall X \in (N \to U).\phi(X)$, that $KPi^+ \vdash \forall x.x \subset \omega \to \phi'(x)$ for the translation ϕ' of ϕ, but only, if we have, that x is an element of the first admissible. (We could easily extend it for x being an element of the first recursive inaccessibel admissible without any trouble, but the result is enough to get an upper bound for the provable proof theoretical strength).)
Anyway, this text serves only to motivate the introduction of non recursive constructors. We have to quantify over all possible choices of new constructors. We will have either constructors, that give as a natural number (functions $\omega^l \to \omega$ for some l), or functions, that gives us an element of the universe, in order to get elements of the powerset of N, but we only need the elements $\underline{n}_0$ (for is not element) and $\underline{n}_1$ (for is an element).

Definition 4.1 *(a) We assume, that we have chosen some Gödel-numbers $\lceil S \rceil$ for all symbols S of ML.*

(b) A tripel $< C, l, f >$ is a constructor definition, if C, l are natural numbers (C is a Gödel number for the constructor), such that C is different from the Gödel-numbers for the symbols, $l > 0$ and f is a function $f : \omega^{l-1} \to \omega$ or $l = 0 \wedge f : \omega \to \{\lceil \underline{n}_0 \rceil, \lceil \underline{n}_1 \rceil\}$. In this situation we define $arity(C) := max\{l, 1\}$.

(c) *A constructor extension set is a finite set of constructor definitions, such that the Gödel-numbers for the constructors are different. We write* $CES(a_0)$ *for* a_0 *being a constructor extension set and* $a_0 \in Ad_2$.

(d) *The* a_0*-extended g-terms, g-types, b-objects are defined as the g-terms, g-types, b-objects, but having on* $arity(C)$ *term constructor for each element* $< C, l, f >$ *of* a_0. $Term_{Cl,a_0}$ *is the set of closed* a_0*-extended t-terms.*

(e) $\forall CES(a_0).\phi(a_0) :\equiv \forall a_0 \in Ad_2.CES(a_0) \to \phi(a_0)$

Assumption 4.2 *As long as there is no other* CES *mentioned, let* a_0 *be a* CES *s.t.* $a_0 \in Ad_2$. *Most of the next definitions will depend on* a_0, *which we will not mention in the following. If we have to mention it, we will add it as a subscript.*

Definition 4.3 (a) *We choose some Gödelization of* a_0*-extended b-terms, but will omit the Gödel-brackets.*

(b) *The introductory term constructors are the term constructors* 0, $\underline{r}$, $\underline{n}$, S, i, j, p, sup, $\widetilde{+}$, π, σ, w, $\widetilde{i}$ *and for* $k \in \omega$, $\underline{n}_k$.

(c) *Let* $\to_{red,imm_{a_0}}$ *or short* $\to_{red,imm}$ *be the relation between closed* a_0*-extended g-terms, defined by*

$p_0(p(r,s)) \to_{red,imm} r$ $\qquad$ $p_1(p(r,s)) \to_{red,imm} s$
$Ap(\lambda x.r, s) \to_{red,imm} r[x/s]$ $\qquad$ $C_n(i_n, r_1, \ldots, r_n) \to_{red,imm} r_i$
$D(i(r), s, t) \to_{red,imm} sr$ $\qquad$ $D(j(r), s, t) \to_{red,imm} tr$
$P(0, s, t) \to_{red,imm} s$ $\qquad$ $P(Sr, s, t) \to_{red,imm} (trP(r, s, t))$
(note that we write rs *for* $Ap(r, s)$*),*
$R(sup(r, s), t) \to_{red,imm} (trs(\lambda z_i.R(sz_i, t)))$, *where* i *is minimal such that* $z_i \notin FV(s) \cup FV(t)$
$C(S^{n_1}0, \ldots, S^{n_l}0) \to_{red,imm_{a_0}} S^{f(n_1,\ldots,n_l)}0$, *(if* $< C, l+1, f > \in a_0$*),*
$C(S^n 0) \to_{red,imm_{a_0}} g(n)$ *(if* $< C, 0, f > \in a_0$, *and* $f : \omega \to \{\underline{n}_0, \underline{n}_1\}$*)*

(d) *We define inductively a set of (indices for) terms in normal-form* $Term_{nf}$, *a subset of the closed* a_0*-extended g-terms:*
If C *is an introductory* n*-ary term constructor,* $t_1, \ldots, t_n \in Term_{nf}$, *then* $C(t_1, \ldots, t_n) \in Term_{nf}$.
If C *is a* n*-ary term constructor (possibly an extended term constructor) that is not introductory,* $t_1, \ldots, t_n \in Term_{nf}$, *and there exists no* t *such that* $C(t_1, \ldots, t_n) \to_{red,imm} t$, *then* $C(t_1, \ldots, t_n) \in Term_{nf}$.
If $t \in Term$, $x \in Var_{ML}$, $FV(t) \subset \{x\}$, *then* $\lambda x.t \in Term_{nf}$.

(e) *We define for* a_0*-extended g-terms* t, *the next reduced term* t^{red}.
For $t \in Term_{nf}$. $t^{red} := t$.
If C *is a* n*-ary (possibly extended) term constructor,* $r_i \in Term_{Cl}$, $\exists i.r_i \notin Term_{nf}$, *then* $C(r_1, \ldots, r_n)^{red} := C(r_1^{red}, \ldots, r_n^{red})$.
If $t := C(r_1, \ldots, r_n) \notin Term_{nf}$, $r_i \in Term_{nf}$, *then* $t \to_{red,imm} t'$ *for some* t', $t^{red} := t'$.

We define $r \to_{red} s$ *iff there exists a sequence* $< s_0, \ldots, s_n >$ *such that* $r = s_0$, $s = s_n$ *and* $\forall i < n.s_{i+1} = (s_i)^{red}$.

Lemma 4.4 (a) $KPi^+ \vdash \forall r, s, s' \in Term_{Cl}.(r \to_{red} s \wedge r \to_{red} s') \to (s \to_{red} s' \vee s' \to_{red} s)$.

(b) $KPi^+ \vdash \forall r, s, s' \in Term_{Cl}.(r \to_{red} s \wedge r \to_{red} s' \wedge s, s' \in Term_{nf}) \to s = s'$.

(c) *If C is a n-ary constructor, then*

$$\begin{aligned} KPi^+ \vdash \quad & \forall r_1, \ldots, r_n, r'_1, \ldots, r'_n.(r_1 \rightarrow_{red} r'_1 \in Term_{nf} \wedge \cdots \wedge r_n \rightarrow_{red} r'_n \in Term_{nf}) \\ & \rightarrow (C(r_1, \ldots, r_n) \rightarrow_{red} C(r'_1, \ldots, r'_n)) \end{aligned}$$

(d) $KPi^+ \vdash \forall t, t', s \in Term_{Cl}.(t \rightarrow_{red} s \wedge t =_\alpha t') \rightarrow \exists s' \in Term_{Cl}.t' \rightarrow_{red} s' \wedge s =_\alpha s'$.

(e) $KPi^+ \vdash \forall t, t' \in Term_{Cl}.t =_\alpha t' \rightarrow (t \in Term_{nf} \leftrightarrow t' \in Term_{nf})$.

Definition 4.5 *If F is a Σ function, we define by recursion on $\alpha \in Ord$*

$$F^\alpha := \begin{cases} \emptyset & \text{if } \alpha = 0, \\ F(F^\beta) & \text{if } \alpha = \beta + 1, \\ \bigcup_{\beta < \alpha, \beta \in Ord} F^\beta & \text{if } \alpha \in Lim. \end{cases}$$

Definition 4.6 (a) *Let $Compl$ be the Σ-function*

$$\begin{aligned} Compl(u) := \quad & \{ < r, s > \in Term_{cl} \times Term_{cl} | \exists r', s' \in Term_{a,nf}. \\ & r \rightarrow_{red} r' \wedge s \rightarrow_{red_a} s' \wedge < r', s' > \in u \} \end{aligned}$$

(b) $N_k^{basis} := \{ < n_k, n_k > | n < k \}$, $N_k^{**} := Compl(N_k^{basis})$, *which are Σ-functions, depending on the parameter k.*

(c) $N^{basis} := \{ < S^n 0, S^n 0 > | n < \omega \}$, $N_k^{**} := Compl(N^{basis})$.

(d)

$$\begin{aligned} \Pi^{basis}(u, f) := \quad & \{ < \lambda x.s, \lambda x'.s' > \in Term_{nf} \times Term_{nf} | \\ & \forall < r, r' > \in u. < s[x/r], s'[x'/r'] > \in f(r) \wedge f(r) = f(r') \} \end{aligned}$$

(more precisely we have to write:

$$\begin{aligned} \Pi^*(u, f) := \quad & \{ < t, t' > \in Term_{nf} \times Term_{nf} | \\ & \exists x, x' \in Var_{ML}, r, r' \in Term.t = \lambda x.s \wedge t' = \lambda x'.s' \wedge \\ & \forall r, r' \in Term_{Cl}. < r, r' > \in u \rightarrow \\ & \quad < s[x/r], s[x'/r'] > \in f(r) \wedge f(r) = f(r') \} \end{aligned}$$

similarly in the following definitions)
$\Pi^*(u, f) := Compl(\Pi^{basis}(u, f))$.

(e)

$$\begin{aligned} \Sigma^{basis}(u, f) := \quad & \{ < p(r, s), p(r', s') > \in Term_{nf} \times Term_{nf} | \\ & < r, r' > \in u \wedge < s, s' > \in f(r) \wedge f(r) = f(r') \} \end{aligned}$$

$\Sigma^*(u, f) := Compl(\Sigma^{basis}(u, f))$.

(f) *Let* $\lambda^*(u) := \{ < t, u > | t \in Term_{Cl} \}$.

(g) $W^*(u, f, \alpha) := F_W(u, f)^\alpha$,
where $F_W(u, f)(v) = Compl(F_W^{basis}(u, f)(v))$, *and*

$$\begin{aligned} F_W^{basis}(u, f)(v) := \quad & \{ < sup(r, s), sup(r', s') > \in Term_{nf} \times Term_{nf} | \\ & < r, r' > \in u \wedge < s, s' > \in \Pi^{basis}(f(r), \lambda^*(v)) \wedge f(r) = f(r') \} \end{aligned}$$

(h)

$$+^{basis}(u,v) := \quad \{< i(r), i(r') >\in Term_{nf} \times Term_{nf} | < r, r' >\in u\} \cup \{< j(r), j(r') >\in Term_{nf} \times Term_{nf} | < r, r' >\in v\}$$

$+^*(u,v) := Compl(+^{basis}(u,v))$.

(i)

$$i^{basis}(u,s,t) := \{< \underline{r}, \underline{r} >\in Term_{nf} \times Term_{nf} | < s,t >\in u\}$$

$i^*(u,s,t,) := Compl(i^{basis}(u,s,t))$.

We will interpret each g-type occurring in a proof of Martin-Löf's type theory as Σ-functions, with arguments represented by the free variables of the type. More precisely, if $FV(A) = \{z_1, \ldots, z_n\}$, ($z_i$ as in the definition 1.1 of Var_{ML}) the arguments of the interpretation A^* will have arguments given by the variables $\{u_1, \ldots, u_n\}$ (u_i as in definition 3.1 (a) of Var_{KP}). We introduce the following abbreviation:

Definition 4.7 *(a) If A is a Σ function in KPi^+, u_i as in the definition 3.1 (a) of Var_{KP} z_i as in the definition 1.1 of Var_{ML}, $r_1, \ldots, r_m$ extended b-objects,*

$$A[z_{i_1}/r_1, \ldots, z_{i_n}/r_n] := A[u_{i_1}/r_{j_1}, \ldots, u_{i_n}/r_{j_n}],$$

where on the right-hand side we have the real substitution in KPi^+ in such a way, that, if a variable occurrs more than once, only the first occurrence is carried out. If we introducing symbols for Σ-functions, this substitution is the application of the Σ-function to the arguments in the ordering as specified by the definition of the function.
We will write $A[\vec{x}/\vec{n}]$ for $A[x_1/n_1, \ldots, x_n/n_n]$.

(b) In the situation as above let $(z_i)A$ be the Σ-function with the same arguments as A except u_i s.t. $((z_i)A)[\vec{x}/\vec{r}] = \{< u, A[z_i/u, \vec{x}/\vec{r}] > | u \in Term_{Cl}\}$.

Definition 4.8 *We define for every g-type A the Sigma-function A^* together with $level(A) \in \omega$.*
If $FV(A) = \{z_1, \ldots, z_n\}$, ($z_i$ as in definition 1.1 of Var_{ML}), A^* will have arguments given by the variables $\{u_1, \ldots, u_n\}$ (u_i as in defition 3.1 (a) of Var_{KP}). We will define it by giving the values $A^*[\vec{x}/\vec{s}]$.
Additionally we define $lev(A) \in \omega$.
For $k \in \omega$, $N_k^*[\,] := N_k^{**}$, $lev(N_k) := 0$.

$N^*[\,] := N^{**}$, $lev(N) := 0$.
Let A, B be g-types, $m := max\{lev(A), lev(B)\}$.
$lev(\Pi x \in A.B) := m$, $(\Pi x \in A.B)^*[\vec{x}/\vec{s}] := \Pi^*(A^*[\vec{x}/\vec{s}], (x)B^*[\vec{x}/\vec{s}])$,
$lev(\Sigma x \in A.B) := m$, $(\Sigma x \in A.B)^*[\vec{x}/\vec{s}] := \Sigma^*(A^*[\vec{x}/\vec{s}], (x)B^*[\vec{x}/\vec{s}])$,
$lev(Wx \in A.B) := m+1$, $(Wx \in A.B)^*[\vec{x}/\vec{s}] := W^*(A^*[\vec{x}/\vec{s}], (x)B^*[\vec{x}/\vec{s}], I_m)$,
$lev(A+B) := m$, $(A+B)^*[\vec{x}/\vec{s}] := +^*(A^*[\vec{x}/\vec{s}], B^*[\vec{x}/\vec{s}])$,
$lev(I(A,s,t)) := level(A)$, $(I(A,s,t))^*[\vec{x}/\vec{s}] := I^*(A^*[\vec{x}/\vec{s}], s[\vec{x}/\vec{s}], t[\vec{x}/\vec{s}])$.
$lev(U) := 1$, $U^*[\,] := \sim(\widehat{U})$,
$lev(T(t)) := 0$, $(T(t))^*[\vec{x}/\vec{s}] := func(\widehat{U})(t[\vec{x}/\vec{s}])$,
where $\widehat{U}$, $\sim(u)$, $func(u)$ will be defined in the next definition.

Definition 4.9 *(a) $\sim(u) := \{< s, s' >\in Term_{Cl} \times Term_{Cl} | \exists v \in TC(u). < s, v, s' >\in u\}$.*

(b) $func(u) := \{< s, v >\in Term_{Cl} \times TC(u) | \exists s' \in Term_{Cl}. < s, v, s' >\in u\}$.

(c) $Compl_U(u) := \{< r, b, r' >\in Term_{Cl} \times TC(u) \times Term_{Cl} | \exists s, s' \in Term_{nf}.$
$r \rightarrow_{red} s \wedge r' \rightarrow_{red} s' \wedge < s, b, s' >\in u\}$.

(d) $\widetilde{U}(u) := Compl_U(\widetilde{U}^{basis}(u))$, *where*

$$\begin{array}{ll} & \widetilde{U}^{basis}(u) := \\ & \{< \underline{n}_k, N_k^{**}, \underline{n}_k >\in ad(u) | k \in \omega\} \\ \cup & \{< \underline{n}, N^{**}, \underline{n} >\} \\ \cup & \{< \pi x \in r.s, \Pi^*(b, f), \pi x' \in r'.s' > | \phi(r, x, s, r', x', s', b, f, u) \wedge b, f \in ad(u)\} \\ \cup & \{< \sigma x \in r.s, \Sigma^*(b, f), \sigma x' \in r'.s' > | \phi(r, x, s, r', x', s', b, f, u) \wedge b, f \in ad(u)\} \\ \cup & \{< wx \in r.s, W^*(b, f, \alpha^+(u)), wx' \in r'.s' > | \\ & \quad \phi(r, x, s, r', x', s', b, f, u) \wedge b, f \in ad(u)\} \\ \cup & \{< r \widetilde{+} s, +^*(b, c), r' \widetilde{+} s' >\in ad(u) | \psi_+(r, s, r', s', b, c, u) \wedge b, c \in ad(u)\} \\ \cup & \{< \widetilde{i}(r, s, t), I^*(b, s, t), \widetilde{i}(r', s', t') >\in ad(u) | \psi_i(r, s, t, r', s', t', b, u) \wedge b \in ad(u)\} \end{array}$$

and

$$\begin{array}{rl} \phi(& r, x, s, r', x', s', b, f, u) \\ := & r, r' \in Term_{nf} \wedge s, s' \in Term \\ & \wedge FV(s) \subset \{x\} \wedge FV(s') \subset \{x'\} \wedge < r, b, r' >\in u \wedge \\ & (\forall < t, t' >\in b. < s[x/t], f(t), s'[x'/t'] >\in u) \end{array}$$

(note that $f(t) = \bigcup\{c \in TC(f) | < t, c >\in f\}$)

$$\psi_+(r, s, r', s', b, c, u) := r, s, r', s' \in Term_{nf} \wedge < r, b, r' >\in u \wedge < s, c, s' >\in u,$$

$$\begin{array}{rcl} \psi_i(r, s, t, r', s', t', b, u) & := & r, s, t, r', s', t' \in Term_{nf} \wedge < r, b, r' >\in u \wedge \\ & & < s, s' >\in b \wedge < t, t' >\in b, \end{array}$$

(e) $\widehat{U} := \widetilde{U}^I$.

5 Properties of the interpretation

Lemma 5.1 *(a)* $\forall v \subset v'. F_W^*(b, f)(v) \subset F_W^*(b, f)(v')$.

(b) $\forall \gamma < \delta. W^*(b, f, \gamma) \subset W^*(b, f, \delta)$

(c) $(b \in a \wedge f \in a \wedge Ad(a)) \rightarrow \forall \gamma > \alpha(a). W^*(b, f, \gamma) = W^*(b, f, \alpha(a))$.

Proof: (a) immediate, (b) follows from (a) by induction on δ.
(c) It is sufficient to show, with $\alpha := \alpha(u)$, $v := W^*(b, f, \alpha)$, that $F_W(b, f)(v) \subset v$. Since $Compl(v) \subset v$ it is sufficient to prove $F_W^{basis}(b, f)(v) \subset v$).
Now, if $< sup(r, t), sup(r', t') >\in F_W^{basis}(b, f)(v)$, then $t = \lambda x.s$, $t' = \lambda x'.s'$, and $\forall u, u' \in Term_{Cl}. < u, u' >\in f(r) \rightarrow \exists \delta \in Ord \cap \alpha. < s[x/u], s'[x'/u'] >\in W^*(b, f, \delta)$
Since $Ad(a)$, we have $(\Delta_0 - coll)^a$, therefore for some $\rho < \alpha$, $\forall t, t' \in Term_{Cl}. < t, t' >\in f(r) \rightarrow \exists \delta < \rho. < s[x/t], s'[x'/t'] >\in W^*(b, f, \delta)$.
Now follows $< sup(r, \lambda x.s),\ sup(r', \lambda x'.s') >\in W^*(b, f, \rho) \subset v$ and the assertion.

Definition 5.2 *(a)* $equiv(u) :\Leftrightarrow$
$\forall r,s,t,r',s' \in Term_{Cl}.(<r,s>\in u \rightarrow <s,r>\in u)\wedge$
$((<r,s>\in u \wedge <s,t>\in u) \rightarrow <r,t>\in u)$.
(note that we do not claim reflexivity)

(b) $equivfun(f) :\Leftrightarrow \forall x \in dom(f).equiv(f(x))$.

(c) $Cor(u) :\Leftrightarrow$
$\forall r,r',r'' \in Term_{Cl}.\forall b,b'.(<r,b,r'>\in u \rightarrow [<r',b,r>\in u \wedge equiv(b)$
$\wedge[<r',b',r''>\in u \rightarrow (<r,b,r''>\in u \wedge b = b'>]])$

Remark 5.3 *(a)* $(Cor(u)\wedge <r,b,r'>\in u\wedge <r,b',r''>\in u) \rightarrow (b = b'\wedge <r,b,r''>\in u)$.

(b) If $Cor(u)$ *then with* $\sim:=\sim(u)$, $f := func(u)$ *we have* $\sim$ *is a symmetric and transitive relation,* f *is a function s.t.* $\forall a,b.a \sim b \rightarrow f(a) = f(b)$ *and* $equivfun(f)$.

Lemma 5.4 *(a)* $(equiv(u) \wedge u \subset Term_{nf} \times Term_{nf}) \rightarrow equiv(Compl(u))$.

(b) $(equiv(u) \wedge equiv(v) \wedge equivfun(f) \wedge k \in \omega \wedge s,t \in Term_{Cl}) \rightarrow$
$(equiv(N^{basis}) \wedge equiv(N^{basis}_k) \wedge equiv(\Pi^{basis}(u,f)) \wedge equiv(\Sigma^{basis}(u,f)) \wedge$
$equiv(F^{basis}_W(u,f)(v)) \wedge equiv(+^{basis}(u,v)) \wedge equiv(i^{basis}(u,s,t)))$.

(c) $(equiv(u) \wedge equiv(v) \wedge equivfun(f) \wedge \alpha \in Ord \wedge k \in \omega \wedge s,t \in Term_{Cl}) \rightarrow$
$(equiv(N^*)\wedge equiv(N^*_k)\wedge equiv(\Pi^*(u,f))\wedge equiv(\Sigma^*(u,f))\wedge equiv(F^*_W(u,f)(v))\wedge$
$equiv(W^*(u,f,\alpha)) \wedge equiv(+^*(u,v)) \wedge equiv(i^*(u,s,t)))$.

Lemma 5.5 *Assume* $r,s,t,r',s',t' \in Term$, $x,x' \in Var_{ML}$, b,f,u *sets.*

(a) $(\phi(r,x,s,r',x',s',b,f,u) \wedge Cor(u)) \rightarrow$
$(b \in ad(u) \wedge \exists f \in ad(u).\forall <t,t'>\in b.f(t) = f(t') = f'(t) = f'(t'))$.

(b) $\psi_+(r,s,r',s',b,c,u) \rightarrow b,c \in ad(u)$.

(c) $\psi_i(r,s,t,r',s',t',b,u) \rightarrow b \in ad(u)$.

(d) $Cor(u) \rightarrow \widetilde{U}(u) \in ad(ad(u))$.

(e) $\forall\gamma \in Ord \cap I.\widetilde{U}^\gamma \in Ad_I$.

Proof:
(a) $b \in TC(u) \in ad(u)$.
Let $f' := \{<t,c>\in Term_{Cl} \times TC(u)| <t,t>\in b\wedge <s[x/t],c,s[x/t]>\in u\}$.
$f' \in ad(u)$. Further, if $<t,t'>\in b$ follows $<s[x/t],f(t),s'[x/t']>\in u$, by $Cor(u)$ $<s[x/t],f(t),s[x/t]>\in u$, $f(t) = f'(t)$, and, since $<s'[x/t],f(t),s[x/t]>\in u\wedge <s'[x/t],f(t'),s[x/t']>\in u$ follows $f(t) = f(t')$.
(b), (c),(d): easy.
(e): Induction on γ, using (d) and 3.3 (b).

Lemma 5.6 *Assume* $r,s,t,r',s',t',r'',s'',t'' \in Term$, $x,x',x'' \in Var_{ML}$, b,b',f,f',u,u' *sets.*

(a) $\phi(r,x,s,r',x',s',b,f,u) \rightarrow Cor(u) \rightarrow \phi(r',x',s',r,x,s,b,f,u)$.

(b) $(\phi(r,x,s,r',x',s',b,f,u) \wedge \phi(r',x',s',r'',x'',s'',b',f',u') \wedge Cor(u \cup u')) \rightarrow$
$\phi(r,x,s,r'',x'',s'',b,f,u \cup u') \wedge b = b'\wedge$
$\forall <t,t'>\in b.f(t) = f'(t) = f(t') = f'(t')$

(c) $\psi_+(r,s,r',s',b,c,u) \to Cor(u) \to \psi_+(r',s',r,s,b,c,u)$.

(d) $(\psi_+(r,s,r',s',b,c,u) \wedge \psi_+(r',s',r'',s'',b',c',u') \wedge Cor(u \cup u') \to$
$(\psi_+(r,s,r'',s'',b,c,u \cup u') \wedge b = b' \wedge c = c'))$

(e) $\psi_i(r,s,t,r',s',t',b,u) \to Cor(u) \to \psi_i(r',s',t',r,s,t,b,u)$.

(f) $(\psi_i(r,s,t,r',s',t',b,u) \wedge \psi_i(r',s',t',r'',s'',t'',b',u') \wedge Cor(u \cup u') \to$
$(\psi_i(r,s,t,r'',s'',t'',b,u \cup u') \wedge b = b'))$

Lemma 5.7 *(a)* $(Cor(u) \wedge \sim (u) \subset Term_{nf} \times Term_{nf}) \to Cor(Compl_U(u))$.

(b) $Cor(u) \to Cor(\widetilde{U}(u))$,

(c) $u \subset u' \wedge Cor(u') \to \widetilde{U}(u) \subset \widetilde{U}(u')$,

(d) $Cor(\widehat{U})$.

Lemma 5.8 *If A g-type, then* $KPi^+ \vdash \forall s_1,\ldots,s_n \in Term_{Cl}.equiv(A^*[\vec{x}/\vec{s}])$

Proof: Induction on the definition of types.

Definition 5.9 *Let A, B g-types, s, t g-terms, $FV(A)$, $FV(B)$, $FV(s)$, $FV(t) \subset \{x_1,\ldots,x_n\}$, $r_1,\ldots,r_n$, $s_1,\ldots,s_n$ be extended g-terms.*

(a) $(A = B)^*[\vec{x}/\vec{r};\vec{s}] :\Leftrightarrow (A = B)^*[x_1/r_1;s_1,\ldots,x_n/r_n;s_n] :\Leftrightarrow (A^*[\vec{x}/\vec{r}] = B^*[\vec{x}/\vec{s}])$.

(b) $(t = t' : A)^*[\vec{x}/\vec{r};\vec{s}] :\Leftrightarrow (t = t' : A)^*[x_1/r_1;s_1,\ldots,x_n/r_n;s_n] :\Leftrightarrow$
$< t[\vec{x}/\vec{r}], t'[\vec{x}/\vec{s}] > \in A^*[\vec{x}/\vec{r}]$.

We will not mention the variables $x_1,\ldots,x_n$ explicitly, if they are the variables, mentioned in the context, writing $(A = B)^*[\vec{r};\vec{s}]$ and $(t = t' : A)^*[\vec{r};\vec{s}]$.
Note that $s : A$, A *type* abbreviate $s = s : A$, $A = A$, therefore $(s : A)^*[\vec{r};\vec{s}]$, $(A\ type)^*[\vec{r};\vec{s}]$ are defined as well.

Lemma 5.10 *(Substitution lemma).*
Let C, D be g-types, r, s, t_i, t_i' g-terms, $x_i, y_i \in Var_{ML}$. Then:

(a) If $r[\vec{x}/\vec{t}]$ is an allowed substitution, $FV(r[\vec{x}/\vec{t}]) \subset \{y_1,\ldots,y_n\}$, then
$KPi^+ \vdash \forall \vec{r} \in Term_{Cl}.r[\vec{x}/\vec{t}][\vec{y}/\vec{r}] = r[x_1/t_1[\vec{y}/\vec{r}],\ldots,x_n/t_n[\vec{y}/\vec{r}],\vec{y}/\vec{r}]$.
(Note that, if variables occur more than once in $[\vec{y}/\vec{r}]$, only the first substitution is relevant.)

(b) If $C[\vec{x}/\vec{t}]$ is an allowed substitution, $FV(C[\vec{x}/\vec{t}]) \subset \{y_1,\ldots,y_n\}$, then
$KPi^+ \vdash \forall \vec{r}, r' \in Term_{Cl}.C[\vec{x}/\vec{t}]^*[\vec{y}/\vec{r}] = C^*[x_1/t_1[\vec{y}/\vec{r}],\ldots,x_n/t_n[\vec{y}/\vec{r}],\vec{y}/\vec{r}]$.

(c) If $A[\vec{x}/\vec{t}]$, $B[\vec{x}/\vec{t'}]$ are allowed substitutions,
$FV(A[\vec{x}/\vec{t}]), FV(B[\vec{x}/\vec{t'}]) \subset \{y_1,\ldots,y_n\}$, then
$KPi^+ \vdash \forall \vec{r}, \vec{s} \in Term_{Cl}.(A = B)^*[\vec{x}/(\vec{t}[\vec{y}/\vec{r}]);(\vec{t'}[\vec{y}/\vec{s}]),\vec{y}/\vec{r};\vec{s}] \leftrightarrow$
$(A[\vec{x}/\vec{t}] = B[\vec{x}/\vec{t'}])^*[\vec{y}/\vec{r};\vec{s}]$.

(d) If $A[\vec{x}/\vec{t}]$, $r[\vec{x}/\vec{t}]$ are allowed substitutions, $FV(A[\vec{x}/\vec{t}]), FV(r[\vec{x}/\vec{t}]) \subset \{y_1,\ldots,y_n\}$, then
$KPi^+ \vdash \forall \vec{r}, \vec{s} \in Term_{Cl}.(r : A)^*[\vec{x}/(\vec{t}[\vec{x}/\vec{r}]);(\vec{t}[\vec{x}/\vec{s}]),\vec{x}/\vec{r};\vec{s}] \leftrightarrow$
$(r[x/t] : A[x/t])^*[\vec{x}/\vec{r};\vec{s}]$.

(e) *If* $A[\vec{x}/\vec{t}]$, $r[\vec{x}/\vec{t}], s[\vec{x}/\vec{t'}]$ *are allowed substitutions,* $FV(A[\vec{x}/\vec{t}])$, $FV(r[\vec{x}/\vec{t}])$, $FV(s[\vec{x}/\vec{t'}]) \subset \{y_1, \ldots, y_n\}$, *then* $KPi^+ \vdash \forall \vec{r}, \vec{s} \in Term_{Cl}.(r = s : A)^*[\vec{x}/(\vec{t}[\vec{x}/\vec{r}]); (\vec{t'}[\vec{x}/\vec{s}]), \vec{x}/\vec{r}; \vec{s}] \leftrightarrow (r[\vec{x}/\vec{t}] = s[x/\vec{t'}] : A[x/\vec{t}])^*[\vec{x}/\vec{r}; \vec{s}]$.

Proof by induction on the definition of the terms and types.

Lemma 5.11 *For every g-type* A $FV(A) \subset \{x_1, \ldots, x_n\}$, *follows*

(a) $\forall \vec{r}, r, r', s, s' \in Term_{Cl}.(r \rightarrow_{red} r') \rightarrow (s \rightarrow_{red} s') \rightarrow < r, s > \in A^*[\vec{x}/\vec{r}] \rightarrow < r', s' > \in A^*[\vec{x}/\vec{r}]$.

(b) $\forall \vec{r}, r, r' \in Term_{Cl}. < r, r' > \in A^*[\vec{x}/\vec{r}] \rightarrow \exists s, s' \in Term_{nf}.r \rightarrow_{red} s \wedge r' \rightarrow_{red} s'$.

Proof: easy, since for each type, *Compl* was applied to some set.

Definition 5.12 (a) $Stable(a) := \forall r, s, r', s' \in Term_{Cl}. < r, s > \in a \rightarrow r =_\alpha r' \rightarrow s =_\alpha s' \rightarrow < r', s' > \in a$

(b) *For every g-type* A *with* $FV(A) = \{x_1, \ldots, x_n\}$ *we define*
$Flex(A) := \forall r_1, \ldots, r_n, s_1, \ldots, s_n \in Term_{Cl}.v(r_1 =_\alpha s_1 \wedge \cdots \wedge r_n =_\alpha s_n) \rightarrow A^*[\vec{x}/\vec{r}] = A^*[\vec{x}/\vec{s}]$

Lemma 5.13 *For every g-type* C, D *with* $FV(C) = \{x_1, \ldots, x_n\}$ *and* $C =_\alpha D$ *we have*

(a) $KPi^+ \vdash Flex(C)$

(b) $KPi^+ \vdash \forall r_1, \ldots, r_n \in Term_{Cl}.Stable(C^*[\vec{x}/\vec{r}])$

(c) $KPi^+ \vdash \forall r_1, \ldots, r_n \in Term_{Cl}.C^*[\vec{x}/\vec{r}] = D^*[\vec{x}/\vec{r}]$.

Proof Easy, simultaneously by induction on the definition of g-types. In the case $C \equiv U, T(t)$ we define

$$\begin{aligned} Stable_U(u) := \quad & \forall s, s', t, t' \in Term_{Cl}.\forall b \in TC(u). \\ & s =_\alpha s' \rightarrow t =_\alpha t' \rightarrow pair(s, b, t) \in u \rightarrow (pair(s', b, t') \in u \wedge Stable(b)) \end{aligned}$$

easily conclude $Stable_U(u) \rightarrow Stable_U(\widetilde{U}(u))$ and therefore $Stable_U(\widehat{U})$, and are now able to prove easily the assertion.

In order to state our Main Lemma, we need to express, that, if we assume elements of the types of the context, the interpretation of the conclusion Θ of a statement of Martin-Löf is valid. Since we need, that this is independent of the choice of equal elements of A_i, we will introduce the following abbreviation:

Definition 5.14 *Let* $\Gamma \equiv x_1 : A_1, \ldots, x_k : A_k$ *be a g-context.*

$$\begin{aligned} \forall \Gamma^=(\vec{r}; \vec{s}).\phi :\equiv \forall r_1, \ldots, r_k, s_1, \ldots, s_k \in Term_{Cl}.(< r_1, s_1 > \in A_1^*[\,] \wedge \\ < r_2, s_2 > \in A_2^*[x_1/r_1] \wedge \cdots \wedge < r_k, s_k > \in A_k^*[x_1/r_1, \ldots, x_{k-1}/r_{k-1}]) \rightarrow \phi \end{aligned}$$

"Assume $\Gamma^=(\vec{r}; \vec{s})$*" means:*
"Assume $r_1, \ldots, r_k, s_1, \ldots, s_k \in Term_{Cl}$ *such that* $< r_1, s_1 > \in A_1^*[\,] \wedge < r_2, s_2 > \in A_2^*[x_1/r_1] \wedge \cdots \wedge < r_k, s_k > \in A_k^*[x_1/r_1, \ldots, x_{k-1}/r_{k-1}]$.

6 Main Lemma

In this section we prove the Main Lemma, which expresses that if $ML \vdash r : A$, then $KPi^+ \vdash < r, r > \in A^*$. We have to go through all statements.

Lemma 6.1 *(***Main lemma***)*
Let Γ, Δ be g-context-pieces, $x, x_i \in Var_{ML}$, A_i, A, B g-types, t, t' g-terms, θ a g-judgement. Assume $\Gamma = x_1 : A_1, \ldots, x_n : A_n$.

(a) If $ML \vdash \Gamma \Rightarrow t = t' : A$, then

(i) $KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(t = t' : A)^*[\vec{x}/\vec{r}; \vec{s}]$.

(ii) $KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A\ type)^*[\vec{x}/\vec{r}; \vec{s}]$.

(b) If $ML \vdash \Gamma \Rightarrow A = A'$, then

$$KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A = A')^*[\vec{x}/\vec{r}; \vec{s}].$$

(c) If $ML \vdash \Gamma, x : A, \Delta \Rightarrow \theta$, then

$$KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A\ type)^*[\vec{x}/\vec{r}; \vec{s}].$$

Proof of the Main Lemma:
We proof simultaneously (a) - (c) by induction on the derivation. We write IH 3 for the Induction-hypothesis for the 3rd premise, etc. IH 3(c) for the Induction-hypothesis (c) for the 3rd premise of the rule etc.
If there is more than one rule of one category (as in the case of $(REFL)$), we refer to them by $(REFL)_1$, $(REFL)_2$, etc.
Let $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, $\Gamma' = y_1 : B_1, \ldots, y_m : B_m$.
If $\vec{r} = r_1, \ldots, r_n$, $i \leq n$, then $\hat{r}_i := r_1, \ldots, r_{i-1}$ ($\hat{r}_1$ is empty).
If θ is $t = t' : A$ or $A = B$, let $\theta' = A\ type$ (the judgement treated in the cases (i) of (a),(b), or which follows from the assertion in (c).
Distinction by the last rule applied.
Assume that lemma is proved for the premisses of a rule, as stated in the definition, weakened by the context Γ.
We treat only some examples of the rules, covering the more complicated ones.
Case $(SYM)_1$ Assume $\Gamma^=(\vec{r}; \vec{s})$. From $< r_i, s_i > \in A_i^*[\hat{r}_i]$ follows $< s_i, r_i > \in A_i^*[\hat{r}_i]$ and by IH (a,ii) $< s_i, r_i > \in A_i^*[\hat{s}_i]$. By IH (a,i) follows $< t[\vec{s}], t'[\vec{r}] > \in A^*[\vec{s}]$, and by IH (a,ii) $A^*[\vec{r}] = A^*[\vec{s}]$, and by 5.8 follows $(t' = t : A)^*[\vec{r}; \vec{s}]$.
(a,ii) follows from IH (a,ii).
Case $(SYM)_2$ Assume $\Gamma^=(\vec{r}; \vec{s})$. As for $(SYM)_1$ we have $< s_i, r_i > \in A_i^*[\hat{s}_i]$, by IH $A^*[\vec{s}] = B^*[\vec{r}]$ and therefore the assertion.
Case (SUB) Assume $\Gamma^=(\vec{r}; \vec{r}')$, $< s_i, s_i' > \in B_i[x/t]^*[\vec{r}, \hat{s}_i]$. Now by lemma 5.10

$$B_i[x/t]^*[\vec{r}, \hat{s}_i] = B_i^*[x/t[\vec{r}, \hat{s}_i], \vec{r}, \hat{s}_i] = B_i^*[x/t[\vec{r}], \vec{r}, \hat{s}_i]$$

By IH 2 (a,i) $< t[\vec{r}], t'[\vec{r}'] > \in A^*[\vec{r}]$, therefore

$$\theta^*[\vec{x}/\vec{r}; \vec{r}', x/t[\vec{r}]; t'[\vec{r}'], \vec{y}/\vec{s}; \vec{s}'],$$

and by lemma 5.10 $\theta[x/t]^*[\vec{x}/\vec{r}; \vec{r}', \vec{s}; \vec{s}']$, similarly for θ'.
Proof for (c): If „$y : B$" in Γ, follows assertion by IH.

If „$y : B$" in $\Gamma'[x/t]$, follows by IH $B^*[\vec{r}, x/\vec{t}[\vec{r}], \hat{s}_i] = B^*[\vec{r'}, x/t[\vec{r'}], \hat{s}'_i]$, and by 5.10 the assertion.

Case $(REPL1)$ Assume $\Gamma^=(\vec{r};\vec{r'})$, $< s_i, s'_i > \in B_i[x/t]^*[\vec{r}, \hat{s}_i]$. We have $< t[\vec{r}], t'[\vec{r'}] > \in A^*[\vec{r}]$. By 5.10 follows $B^*_i[\vec{x}/\vec{r}, x/t[\vec{r}], \hat{y}_i/\hat{s}_i] = B_i[x/t]^*[\vec{r}, \hat{s}_i]$. Therefore we have $< s_i, s'_i > \in B^*_i[\vec{x}/\vec{r}, x/t[\vec{r}], \hat{y}_i/\hat{s}_i]$ Then by IH 1 $B^*[\vec{x}/\vec{r}, x/t[\vec{r}], \vec{y}/\vec{s}] = B^*[\vec{x}/\vec{r'}, x/t'[\vec{r'}], \vec{y}/\vec{s'}]$, and by 5.10 follows the assertion.

Proof for (c): From IH 2 follows, arguing as for the rule $(REFL)$, the assertion for $\Gamma \Rightarrow t' = t : A$ and further, arguing as for $(TRANS)$ the assertion for $\Gamma \Rightarrow t = t : A$, which is the same as for $\Gamma \Rightarrow t : A$ and now the proof follows as in (SUB).

Case $(REPL2)$ Assume $\Gamma^=(\vec{r},\vec{r'})$, $< s_i, s'_i > \in B_i[x/t]^*[\vec{r}, \hat{s}_i]$. Then $< t[\vec{r}], t'[\vec{r'}] > \in A^*[\vec{r}]$, and by IH 1(a,i)

$$(s = s : B)^*[\vec{x}/\vec{r};\vec{r'}, x/t[\vec{r}]; t'[\vec{r'}], \vec{y}/\vec{s};\vec{s'}]$$

and by 5.10 follows the assertion for (a,i). (a,ii) follows as in $(REPL1)$, using that we get the assertion for $\Gamma \Rightarrow t = t : A$, and (c) follows exactly as in $(REPL1)$.

Case $(ALPHA)$: Immediate by the IH since if $A =_\alpha A'$, $t =_\alpha t'$, $A[\vec{s}] = A'[\vec{s}]$, $t[\vec{r'}] =_\alpha t'[\vec{r'}]$ and $< t[\vec{r}], t[\vec{r'}] > \in A^*[\vec{r}] \leftrightarrow < t[\vec{r}], t'[\vec{r'}] > \in A^*[\vec{r}]$.

Case $(\Pi^{T,=})$ Assume $\Gamma^=(\vec{r};\vec{r'})$. By IH (c) $A^*[\vec{r}] = A'^*[\vec{r'}]$, and, if $< r, s > \in A^*[\vec{r}]$, follows $< r, r >, < s, s > \in A^*[\vec{r}]$, therefore by IH $B^*[\vec{r}, x/r] = B'^*[\vec{r'}, x/r]$, $B^*[\vec{r}, x/s] = B'^*[\vec{r'}, x/s]$, $\Pi x \in A.B^*[\vec{r}] = (\Pi x \in A'.B')^*[\vec{r'}]$.

Case $(N^{I,=})_2$: Assume $\Gamma^=(\vec{r};\vec{r'})$. By IH we have for some $k \in \omega$ $t[\vec{r}] \rightarrow_{red} S^k 0$ and $t'[\vec{r'}] \rightarrow_{red} S^k 0$, therefore $St[\vec{r}] \rightarrow_{red} S^{k+1} 0$, $St'[\vec{r'}] \rightarrow_{red} S^{k+1} 0$, and we have the assertion.

Case $(\Pi^{I,=})$: Assume $\Gamma^=(\vec{r};\vec{r'})$, $< r, r' > \in A^*[\vec{r}]$. Then by IH (a,i)
$< t[x/r, \vec{r}], t'[x/r', \vec{r'}] > \in B^*[x/r, \vec{r}] = B^*[\vec{r}, x/r]$, $< (\lambda x.t)[\vec{x}/\vec{r}], (\lambda x.t')[\vec{x}/\vec{r'}] > \in (\Pi x \in A.B)^*[\vec{r}]$.
(a,ii) follows as in $(\Pi^{T,=}_1)$, since from IH (a,ii) follows (b) for $x : A \Rightarrow B$ *type*.

Case $(W^{I,=})$: Let $n := max\{level(A), level(B)\}$, $Wx \in A.B^*$.
Assume $\Gamma^=(\vec{r};\vec{r'})$. Let $F := F_W(A^*[\vec{x}/\vec{r}], (x)B^*[\vec{x}/\vec{r}])$. Then by IH $< r[\vec{r}], r'[\vec{r'}] > \in A^*[\vec{r}]$, $s[\vec{r}] \rightarrow_{red} \lambda x.t$, $s'[\vec{r'}] \rightarrow_{red} \lambda x'.t'$, $B^*[x/r[\vec{r}], \vec{r}] = B^*[x/r'[\vec{r'}], \vec{r'}]$, and

$$\forall < u, u' > \in B[x/t]^*[\vec{x}/\vec{r}] (= B^*[x/t[\vec{r}], \vec{x}/\vec{r}]) \exists \gamma < I_n. < t[x/u], t'[x/u'] > \in F^\gamma$$

By $(\Delta_0 - coll)$ and $Ad(L_{I_n})$ there exist a $\delta < I_n$ such that the γ can be chosen to be $< \delta$. Then $< sup(r, s)[\vec{r}], sup(r, s)[\vec{r'}] \in F^{\gamma+1} \subset Wx \in A.B^*[\vec{r}]$.
(a,ii) follows as in $(W^{T,=})$.

Case$(N^{E,=})$: Assume $\Gamma^=(\vec{r};\vec{r'})$. Then by IH 1 $< r[\vec{r}], r'[\vec{r'}] > \in N^*$, therefore $r[\vec{r}] \rightarrow_{red} S^n 0$, $r'[\vec{r}] \rightarrow_{red} S^n 0$ for some $n < \omega$. Further by IH 2 and 5.11 (b) exist $\widetilde{s}$, $\widetilde{s}' \in Term_{nf}$ such that $s[\vec{r}] \rightarrow_{red} \widetilde{s}$, $t[\vec{r}] \rightarrow_{red} \widetilde{s}'$, $< \widetilde{s}, \widetilde{s}' > \in A[x/0]^*[\vec{r}] = A^*[x/u, \vec{r}]$.
Let $[\vec{x'}/\vec{s}] := [\vec{x}/\vec{r}] \setminus \{x, y\}$, $[\vec{x'}/\vec{s''}] := [\vec{x}/\vec{r'}] \setminus \{x, y\}$.
Let $P_0(r) := P(r, \widetilde{s}, \lambda x.\lambda y.(t[\vec{x'}/\vec{s}]))$. $P_1(r) := P(r, \widetilde{s}', \lambda x.\lambda y.(t'[\vec{x'}/\vec{s'}]))$. Then

$$P(r, s, (x, y)t)[\vec{r}] \rightarrow_{red} P_0(S^n 0),\ P(r', s', (x, y)t')[\vec{r'}] \rightarrow_{red} P_1(S^n 0)$$

We have $A[z/r]^*[\vec{r}] = A^*[z/r[\vec{r}], \vec{r}] = A^*[z/S^n 0, \vec{r}]$, and therefore assertion (a,i).
We show: $\forall m \in \omega. < P_0(S^m 0), P_1(S^m 0) > \in A^*[z/S^m 0, \vec{r}]$.
If $m = 0$, $P_0(S^m 0) \rightarrow_{red} \widetilde{s}$, $P_1(S^m 0) \rightarrow_{red} \widetilde{s}'$, $< \widetilde{s}, \widetilde{s}' > \in A^*[z/0, \vec{r}]$.
If $m = k + 1$, follows by IH $P_0(S^k 0) \rightarrow_{red} \widetilde{s}$, $P_1(S^k 0) \rightarrow_{red} \widetilde{s}'$,
$\widetilde{s}, \widetilde{s}' \in Term_{nf}$, $< \widetilde{s}, \widetilde{s}' > \in A^*[z/S^k 0, \vec{r}] = A[z/x]^*[x/S^k 0, \vec{r}]$.

$P_0(S^m0)\rightarrow_{red}t[\vec{x}'/\vec{s},x/S^k0,y/\tilde{s}]$, $P_1(S^m0)\rightarrow_{red}t'[\vec{x}'/\vec{s}'',x/S^k0,y/\tilde{s}']$.
Now $< S^k0, S^k0 >\in N^*$, therefore by IH 3 follows

$$< t[\vec{x}'/\vec{s},x/S^k0,y/\tilde{s}], t'[\vec{x}'/\vec{s}'',x/S^k0,y/\tilde{s}'] >\in A[z/Sx]^*[\vec{r}] = A^*[z/S^m0,\vec{r}],$$

and the side induction is finished.
(a,ii) is easy.
Case($\Pi^{E,=}$): Assume $\Gamma^=(\vec{r};\vec{r}')$. By IH 1,2 there exist $\tilde{r},\tilde{r}'\in Term_{nf}$ such that $r[\vec{r}]\rightarrow_{red}\tilde{r}_1$, $r'[\vec{r}']\rightarrow_{red}\tilde{r}'$ $<\tilde{r}_1,\tilde{r}'>\in A^*[\vec{r}]$, and there are $t,t'\in Term$ and Variables $x,x'\in Var_{ML}$ such that

$$s[\vec{r}]\rightarrow_{red}\lambda x.r,\ s'[\vec{s}]\rightarrow_{red}\lambda x'.r',\ <\lambda x.r,\lambda x'.r'>\in(\Pi x\in A.B)^{basis}[\vec{r}].$$

Therefore

$$Ap(s,r)[\vec{r}]\rightarrow_{red}Ap(\lambda x.t,\tilde{r})\rightarrow_{red}t[x/\tilde{r},\vec{r}],$$
$$Ap(s',r')[\vec{r}']\rightarrow_{red}t'[x'/\tilde{r},\vec{r}']$$
$$< t[x/\tilde{r},\vec{r}], t'[x'/\tilde{r}',\vec{r}'] >\in B^*[x/\tilde{r},\vec{r}].$$

As before we conclude

$$< r[\vec{r}], r[\vec{r}] >\in A^*[\vec{r}]$$
$$<\tilde{r}, r_1[\vec{r}] >\in A^*[\vec{r}]$$
$$B^*[x/\tilde{r},\vec{r}] = B^*[x/r[\vec{r}],\vec{r}] = B[x/r]^*[\vec{r}],$$

and we have IH (a,i).
(a,ii) follows as before
Case($\Sigma^{E,=}$): Assume $\Gamma^=(\vec{r};\vec{r}')$. By IH 1 exist $s,s',t,t'\in Term_{nf}$ such that

$$r[\vec{r}]\rightarrow_{red}p(s,t),\ r'[\vec{r}']\rightarrow_{red}p(s',t'),\ < s,s' >\in A^*[\vec{r}],\ < t,t' >\in B^*[x/s,\vec{r}].$$

Then $p_0(r[\vec{r}])\rightarrow_{red}s$, $p_0(r'[\vec{r}'])\rightarrow_{red}s'$, and we are done for the first rule, and $p_1(r[\vec{r}])\rightarrow_{red}t$, $p_1(r'[\vec{r}'])\rightarrow_{red}t'$, and since from $< s,s' >\in A[\vec{r}]$, follows

$$< s,s' >\in A[\vec{r}],\ < p_0(r)[\vec{r}], s >\in A[\vec{r}],$$

therefore by IH 2

$$B^*[x/s,\vec{r}] = B^*[x/p_0(r)[\vec{r}],\vec{r}] = B[x/p_0(r)]^*[\vec{r}]$$

follows (a,i) for the second rule.
(a,ii) is in $(\Sigma^{E,=})_1$ trivial, in $(\Sigma^{E,=})_2$ we use the proof of $(\Sigma^{E,=})_1$ and argue as before.
Case($W^{E,=}$): Assume $\Gamma^=(\vec{r};\vec{r}')$, $n := max\{level(A), level(B)\}$, $F := F_W(A^*[\vec{r}],(x)B^*[\vec{r}])$. By IH $r[\vec{r}]\rightarrow_{red}\tilde{r}$, $r'[\vec{r}']\rightarrow_{red}\tilde{r}'$, $<\tilde{r},\tilde{r}'>\in F^\delta(\vec{r},\cdot)$. for some $\delta<\alpha$. Let

$$[\vec{x}'/\vec{s}] := [\vec{x}/\vec{r}]\setminus\{x,y,z\},$$
$$[\vec{x}'/\vec{s}'] := [\vec{x}/\vec{r}']\setminus\{x,y,z\},$$
$$R_0(r) := R(r,(x,y,z)t)[\vec{r}](= R(r,\lambda x.\lambda y.\lambda z.(t[\vec{s}])))$$
$$R_1(r) := R(r,(x,y,z)t')[\vec{r}'].$$

We show by induction on γ,

$$(+)\qquad \forall\gamma<\alpha.\forall<\tilde{s},\tilde{s}'>\in F^\gamma.<R_0(\tilde{s}),R_1(\tilde{s}')>\in C^*[u/\tilde{s},\vec{r}]$$

Since $C[u/t]^*[\vec{r}] = C^*[u/r[\vec{r}], \vec{r}] = C^*[u/\tilde{r}, \vec{r}] = C^*[u/\tilde{r}', \vec{r}']$ (using arguments as before), follows the assertion.
The case $\gamma = 0$ is trivial, and if $\gamma \in Lim$ follows the assertion by IH
Let now
$\gamma = \gamma' + 1$, $u' := F^{\gamma'}$, $< \tilde{s}, \tilde{s}' > \in F(u')$.
If $\tilde{s} \to_{red} s$, $\tilde{s}' \to_{red} s'$, $< s, s' > \in F^{basis}(\vec{r}', \cdot)$, $< R_0(s), R_1(s') > \in C^*[u/s, \vec{r}]$, follows $< R_0(\tilde{s}), R_1(\tilde{s}') > \in C^*[u/s, \vec{r}]$, further, like similar arguments before,

$$C^*[u/s, \vec{r}] = C^*[u/\tilde{s}, \vec{r}] = C^*[u/\tilde{s}'].$$

We therefore assume $< \tilde{s}, \tilde{s}' > \in F^{basis}(\vec{r}, u')$.
Let $< \tilde{s}, \tilde{s}' > = < sup(a, \lambda x.s), sup(a', \lambda x'.s') >$, $< a, a' > \in A^*[\vec{r}]$. Let $< r'', r''' > \in B^*[x/a, \vec{r}]$. Then $r'' \to_{red} b$, $r''' \to_{red} b'$ for $< b, b' > \in B^*[x/a, \vec{r}]$, $b, b' \in Term_{nf}$, and we have $< s[x/r''], s'[x'/r'''] > \in u'$ and

$$(*) \qquad < s[x/b], s'[x'/b'] > \in u'$$

Since $u' \subset (Wx \in A.B)^*[\vec{r}]$ follows from the first of these assertions

$$< \lambda x.s, \lambda x'.s' > \in (B \to Wx \in A.B)^*[\vec{r}]$$

Further, for $< b, b' > \in B^*[x/r, \vec{r}]$,

$$(R_0((\lambda x.s)v))[v/r''] \to_{red} R_0(s[x/b])(v \notin FV(\lambda x.s))$$

$$(R_1((\lambda x.s')v'))[v'/r'''] \to_{red} R_1(s'[x/b'])(v' \notin FV(\lambda x.s))$$

and by side IH, follows

$$\begin{aligned} &< (R_0((\lambda x.s)v))[v/r''], (R_1((\lambda x'.s')v))[v'/r'''] > \in C^*[u/(s[x/b]), \vec{r}] \\ &\quad = C^*[u/(s[x/r'']), \vec{r}] \end{aligned}$$

Now we have $< r_i, r_i > \in A_i[\hat{r}_i]$, $Ap(\lambda x.s, r'') \to_{red} s[x/b]$, and by $(*)$, $u' \subset (Wx \in A.B)^*[\vec{r}]$, $equiv((Wx \in A.B)^*[\vec{r}])$ and 5.11 follows

$$< s[x/b]], Ap(\lambda x.s, r'') > \in (Wx \in A.B)^*[\vec{r}]$$

therefore

$$C[u/Ap(y, v)]^*[v/r'', y/\lambda x.s, \vec{x}/\vec{r}] = C^*[u/Ap(\lambda x.s, r''), \vec{r}] = C^*[u/(s[x/b]), \vec{r}]$$

further

$$C[u/Ap(y, v)]^*[v/r'', y/\lambda x.s, \vec{x}/\vec{r}] = C[u/Ap(y, v)]^*[v/r''', y/\lambda x.s, \vec{x}/\vec{r}],$$

and we have

$$< \lambda v.R_0((\lambda x.s)v), \lambda v'.R_1((\lambda x'.s')v') > \in (\Pi v \in B.C[u/Ap(y, v)])^*[y/\lambda x.s, \vec{x}/\vec{r}]$$

Now by IH 2 follows

$$\begin{aligned} &< t[x/r, y/\lambda x.s, z/\lambda v.R_0((\lambda x.s)v), \vec{r}], t'[x/r', y/\lambda x'.s', z/\lambda v'.R_1((\lambda x.s')v'), \vec{r}'] > \\ &\qquad \in C[u/sup(x, y)]^*[x/r, y/\lambda x.s, \vec{r}] \end{aligned}$$

Since

$$C[u/sup(x,y)]^*[x/r,y/\lambda x.s,\vec{r}] = C^*[u/sup(r,\lambda x.s),\vec{r}] = C^*[u/s,s],$$

and

$$R_0(s)\rightarrow_{red}(\lambda x.\lambda y.\lambda z.t[\vec{x}'/\vec{s}])r(\lambda x.s)(\lambda v.R_0((\lambda x.s)v))$$
$$\rightarrow_{red}t[x/r,y/\lambda x.s,z/(\lambda v.R_0((\lambda x.s)v))]$$

$$R_1(s')\rightarrow_{red}t'[x/r',y/\lambda x.s',z/(\lambda v'.R_1((\lambda x'.s')v'))]$$

follows (+), and we are done. (a,ii) follows as in the case $(N_k^{E,=})$.
Case $(+^{E,=})$: Assume $\Gamma^=(\vec{r};\vec{r}')$. By IH $r[\vec{r}]\rightarrow_{red}i(\tilde{r}) \in Term_{nf}$, $r'[\vec{r}']\rightarrow_{red}i(\tilde{r}') \in Term_{nf}$ and $<\tilde{r},\tilde{r}'>\in A^*[\vec{r}]$ or $r[\vec{r}]\rightarrow_{red}j(\tilde{r}) \in Term_{nf}$, $r'[\vec{r}']\rightarrow_{red}j(\tilde{r}') \in Term_{nf}$ and $<\tilde{r},\tilde{r}'>\in B^*[\vec{r}]$. Let $[\vec{x}'/\vec{s}] := [\vec{x}/\vec{r}]\setminus\{x\}$. In the first case we have

$$D(r,(x)s,(y)t)[\vec{r}]\rightarrow_{red}(\lambda x.(s[\vec{s}]))\tilde{r}\rightarrow_{red}s[x/\tilde{r},\vec{x}/\vec{r}],$$

$$D(r',(x)s',(y)t')[\vec{r}]\rightarrow_{red}s'[x/\tilde{r}',\vec{x}/\vec{r}'],$$
$$<s[x/\tilde{r},\vec{r}],s'[x/\tilde{r}',\vec{r}']>\in C[z/i(x)]^*[x/\tilde{r},\vec{r}] = C^*[z/i(\tilde{r}),\vec{r}]$$

and using arguments as before

$$C^*[z/i(\tilde{r}),\vec{r}] = C^*[z/r[\vec{r}],\vec{r}] = C[z/r]^*[\vec{r}]$$

and we are done. The second assertion follows in the same way.
(a,ii) follows as before.
Case (I^E): Assume $\Gamma^=(\vec{r};\vec{r}')$. By IH 1 follows $(I(A,s,t))^*[\vec{r}]\neq\emptyset$, $<s[\vec{r}],t[\vec{r}]>\in A^*[\vec{r}]$. Further by IH 3 $<t[\vec{r}],t[\vec{r}']>\in A^*[\vec{r}]$, and by $equiv(A^*[\vec{r}])$ follows (a,i). (a,ii) is trivial.
Case $(\Pi^=)$, $(\Sigma_0^=)$, $(\Sigma_1^=)$: By using the proof for the elimination rules we see, that if the conclusion is $r=s:C$, we conclude assuming $\Gamma^=(\vec{r};\vec{r}')$, that $(r=r:C)[\vec{r};\vec{r}']$, further $(r[\vec{r}']\rightarrow_{red}t\in Term_{nf})\rightarrow(s[\vec{r}']\rightarrow_{red}t)$, therefore follows $(r=s:C)[\vec{r};\vec{r}']$.
Case (Π^η):Assume $\Gamma^=(\vec{r},\vec{r}')$ By IH we have

$$<t[\vec{r}],t[\vec{r}']>\in(\Pi x\in A.B)^*[\vec{r}],$$

therefore $t[\vec{r}]\rightarrow_{red}\lambda x.s$, $t[\vec{r}']\rightarrow_{red}\lambda x'.s'$,

$$<\lambda x.s,\lambda x'.s'>\in(\Pi x\in A.B)^{basis}[\vec{r}],$$

Assume $<r,r'>\in A^*[\vec{r}]$. Then $r\rightarrow_{red}\tilde{r}$, $r'\rightarrow_{red}\tilde{r}'$, $<\tilde{r},\tilde{r}'>\in A^*[\vec{r}]$, $\tilde{r},\tilde{r}'\in Term_{nf}$.

$$Ap(t,x))[\vec{r}][x/r] = Ap(t[\vec{r}],r)\rightarrow_{red}Ap(\lambda x.s,\tilde{r})\rightarrow_{red}s[x/\tilde{r}],$$

and since

$$<s[x/\tilde{r}],s'[x'/\tilde{r}']>\in B^*[x/\tilde{r},\vec{r}] = B^*[x/r,\vec{r}],$$

follows

$$<Ap(t,x)[\vec{r}][x/r],s'[x'/\tilde{r}']>\in B^*[x/r,\vec{r}],$$

therefore

$$<\lambda x.Ap(t,x)[\vec{r}],\lambda x'.s'>\in(\Pi x\in A.B)^*[\vec{r}],$$
$$<\lambda x.Ap(t,x)[\vec{r}],t[\vec{r}]>\in(\Pi x\in A.B)^*[\vec{r}].$$

Case $(\Sigma_2^=)$: Assume $\Gamma^=(\vec{r};\vec{r}')$. By IH $t[\vec{r}]\rightarrow_{red}r$, $t[\vec{r}']\rightarrow_{red}r'$ for some $<r,r'>\in\Sigma x\in$

$A.B^*[\vec{r}] \cap (Term_{nf} \times Term_{nf})$, $p(p_0(t), p_1(t))[\vec{r}] \rightarrow_{red} r$ and we are done.

Case $(I^=)$: Assume $\Gamma^=(\vec{r}; \vec{r}')$. By IH we conclude $< t_0[\vec{r}], t_0[\vec{r}'] > \in I(A, t_1, t_2)^*[\vec{r}]$, therefore $t_0[\vec{r}'] \rightarrow_{red} \underline{r}$, $< \underline{r}, \underline{r} > \in I(A, t_1, t_2)^*[\vec{r}]$, $< t_0[\vec{r}], \underline{r} > \in I(A, t_1, t_2)^*[\vec{r}]$. (a,ii) is trivial.

Case other equality rules: Let $\widetilde{r} = \widetilde{s} : A$ be the conclusion of the rules. By using several times the rules general rules, elimination rules and in case $W^=$ the introduction rules we can conclude $\widetilde{r} = \widetilde{r} : A$, and $\widetilde{s} = \widetilde{s} : A$. (For $(W^=)$ we argue that $\Gamma, v : B[x/t_0] \Rightarrow Ap(t_1, v) : Wx \in A.B$, by $(W^{E,=})$ $\Gamma, v : B[x/t_0] \Rightarrow R(Ap(s', v), (x, y, z)t') : C[u/Ap(s', v)]$, by $\Pi^{I,=}$ $\Gamma \Rightarrow \lambda v.R(Ap(s', v), (x, y, z)t') : \Pi v \in B.C[u/Ap(s', v)]$, by $(ALPHA)$ for the z_i, that we need, and it follows $\Gamma \Rightarrow \lambda z_i.R(Ap(s', z_i), (x, y, z)t') : \Pi v \in B.C[u/Ap(s', v)]$, and now by (SUB) follows the assertion). Now, assuming $\Gamma^=(\vec{r}, \vec{r}')$, and using the proofs above we can conclude $< r[\vec{r}], r[\vec{r}'] > \in A^*[\vec{r}]$ and $A^*[\vec{r}] = A^*[\vec{r}']$, so (a,i). In all the cases, we have, if the right side is written as $t[x_1/r_1, \ldots, x_n/t_n]$, if x_i corresponds to the type B_i (read off from the rule) follows easily by IH and using the proofs of several rules handled before the assertion for $\Gamma \Rightarrow r_i : B_i$, therefore $r_i[\vec{r}] \rightarrow_{red} \widetilde{r}_i \in Term_{nf}$ for some $\widetilde{r}_i$, $< \widetilde{r}_i, r_i[\vec{r}] > \in B_i[\vec{r}]$, further $\widetilde{r}[\vec{r}] \rightarrow_{red} t[x_1/\widetilde{r}_1, \cdots x_n \widetilde{r}_n, \vec{r}]$. We conclude

$$< t[x_1/\widetilde{r}_1, \ldots, x_n/\widetilde{r}_n, \vec{r}], t[x_1/r_1[\vec{r}], \ldots, x_n/r_n[\vec{r}], \vec{r}] > \in A^*[\vec{r}].$$

Now using $equiv(A^*[\vec{r}])$ and lemma 5.11 we conclude

$$< \widetilde{r}[\vec{r}], \widetilde{s}[\vec{r}] > \in A^*[\vec{r}], < \widetilde{s}[\vec{r}], \widetilde{s}[\vec{r}'] > \in A^*[\vec{r}],$$

and have (a,i).

Case (U^I): trivial.

$(T^{I,=})$ we have by IH, assuming $\Gamma^=(\vec{r}; \vec{r}')$,

$$< a[\vec{r}], a'[\vec{r}'] > \in U^*$$

therefore,

$$< a[\vec{r}], b, a'[\vec{r}'] > \in \widehat{U}$$

for some b, by $Cor_U(\widehat{U})$,

$$< a[\vec{r}], b, a[\vec{r}] > \in \widehat{U}, < a[\vec{r}'], b, a[\vec{r}'] > \in \widehat{U},$$

and

$$T(a)^*[\vec{r}] = b = T(a')^*[\vec{r}']$$

Case $(\pi^{I,=})$: Assume $\Gamma^=(\vec{r}; \vec{r}')$. By IH $a[\vec{r}] \rightarrow_{red} \widetilde{a}$, $a'[\vec{r}'] \rightarrow_{red} \widetilde{a}'$,

$$\exists \gamma < I \exists b' \in \ TC(\widetilde{U}^\gamma)(< \widetilde{a}, b', \widetilde{a} >, < \widetilde{a}', b', \widetilde{a} > \in \widetilde{U}^\gamma),$$

and

$$\forall < t, t' > \in b' \rightarrow \exists \delta < I. \exists c \in \ TC(\widetilde{U}^\delta).$$
$$(< b[x/t, \vec{r}], c, b'[x/t', \vec{r}'] > \in \widetilde{U}^\delta).$$

Since $Ad(L_I)$ (here is the central point where we need (Δ_0-coll) and an admissible a which is closed under the step to the next admissible), and $TC(\widetilde{U}^\beta) \in L_I$ ($\beta < I$), there is a $\rho < I$, such that $\gamma < \rho$ and δ can be chosen $< \rho$. There are now b, f such that ($[\vec{x}'/\vec{s}] := [\vec{x}/\vec{r}] \setminus \{x\}$, $[\vec{x}'/\vec{s}'] := [\vec{x}/\vec{r}'] \setminus \{x\}$) $\phi(\widetilde{a}, x, b[\vec{s}], \widetilde{a}', x, b[\vec{s}'], b, f, \widetilde{U}^\rho)$, (note that the c we used above is correct by $Cor(\widehat{U})$) and by 5.5 (a) follows

$$< \pi x \in a.b, \Pi^*(b, f), \pi x \in a'.b' > \in \widetilde{U}^{\rho+1}.$$

Case ($\pi^=$): Assume $\Gamma^=(\vec{r};\vec{r'})$ and chose b', f, ρ as in $(\pi^{I,=})$. Then $T(a)^*[\vec{r'}] = b'$, and if $< t, t' >\in b'$, $T(b)^*[x/t, \vec{r}] = f(t) = f(t') = T(b)^*[x/t', \vec{r'}]$. Since we have $Cor(\widehat{U})$ (by lemma **?? ??**) follows

$$T((\pi x \in a.b))^*[\vec{r}] = \Pi^*(T(a)[\vec{r}], (x)T(b)[\vec{r}]) = (\Pi x \in T(a).T(b))^*[\vec{r}].$$

In the case of $(w^=)$ we conclude as before, that

$$F_W(T(a)^*[\vec{r}], (x)T(b)^*[\vec{r}]) = F_W(T(a)^*[\vec{r'}], (x)T(b)^*[\vec{r'}])$$

and, since
$\alpha^+(\widetilde{U}^\rho) < I$, ($\rho$ chosen as in $(\pi^{I,=})$) follows by 5.1
$T(wx \in a.b)^*[\vec{r}] = F_W^{\alpha^+(\widetilde{U}^\rho)}(T(a)^*[\vec{r}], (x)(T(b))^*[\vec{r}]) = F_W^I(T(a)^*[\vec{r}], (x)(T(b))^*[\vec{r}]) =$
$(Wx \in T(a).T(b))^*[\vec{r'}]$
Case (σ^E): Assume $\Gamma^=(\vec{r};\vec{r'})$. By IH exists c, $\alpha \prec I$ such that

$$< (\sigma x \in a.b)[\vec{r}], c, (\sigma x \in a.b)[\vec{r'}] >\in \widetilde{U}^\alpha.$$

Let α be chosen minimal. Then, $\alpha = \alpha' + 1$, and with $u := \widetilde{U}^{\alpha'}$ there exist $r, r' \in Term_{nf}$, c, c', f, f' such that $a[\vec{r}]\rightarrow_{red} r$, $a[\vec{r'}]\rightarrow_{red} r'$, $< r, c, r' >\in u$ and (with $[\vec{x'}/\vec{s}] := [\vec{x}/\vec{r}] \setminus \{x\}$, $[\vec{x'}/\vec{s'}] := [\vec{x}/\vec{r'}] \setminus \{x\}$) $\forall < t, t' >\in c.\ < s[\vec{s}][x/t], f(t), s[\vec{s'}][x/t'] >\in u$. Therefore $T(a[\vec{r}])^* = c = T(a[\vec{r'}])^*$, and for $< t, t' >\in T(a[\vec{r}])^*$, $< s[x/t, \vec{r}], s[x/t', \vec{s'}] >\in U^*[\ ]$.
(π^E), (w^E), $(\widetilde{+}^E)$ are checked in the same way. For $(\widetilde{i}^E)$ we observe, that $a[\vec{r}]\rightarrow_{red}\widetilde{a}$, $a[\vec{r'}]\rightarrow_{red}\widetilde{a}'$, $s[\vec{r}]\rightarrow_{red}\widetilde{s}$, $s[\vec{r'}]\rightarrow_{red}\widetilde{s}'$, $t[\vec{r}]\rightarrow_{red}\widetilde{t}$, $t[\vec{r'}]\rightarrow_{red}\widetilde{t}'$, and $< \widetilde{a}, c, \widetilde{a}' >\in u$, for some u as before, $T(a)^*[\vec{r}] = c$, $< \widetilde{s}, \widetilde{s}' >\in c$, $< \widetilde{t}, \widetilde{t}' >\in c$, and since c is closed under $\rightarrow_{red}$ follows the assertion.

7 Arithmetical formulas in ML and KPi^+

In this section we want to evaluate the results we have found out to get the proof theoretical strength of Martin-Löf's type theory. We will interpret the language of analysis ($L_{analysis}$, introduced in 7.1) in L_{ML} and L_{KP} (definition7.5) and prove that it permutes with the interpretation of Martin-Löf's type theory in KPi^+ (lemmata 7.7 and 7.9). Next we observe, that we could interpret every proof in KPi^+, ???? and have a stronger bound (lemma **??**). At the end we analyze the proof theoretical strength of KPi^+_n and have the desired upper bound (theorem 7.14).

Definition 7.1 *Definition of the language of Peano Arithmetic $L_{analysis}$: we have first order variables v_i ($i \in \omega$, $var_{analysis} := \{v_i | i \in \omega\}$); second order variables V_i ($i \in \omega$, $VAR_{analysis} := \{V_i | i \in \omega\}$); further we have symbols for each primitive recursive function, $=$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, $\perp$, and $.,,,(,)$.*
Terms are first-order variables and $f(t_1, \ldots, t_n)$ if t_i are terms and f is a symbol for a n-ary primitive recursive function.
Prime formulas are $\perp$, equations $r = s$, and $r \in X$ for r, s terms, $X \in VAR_{analysis}$.
Formulas are prime formulas and $A \rightarrow B$, $A \wedge B$, $A \vee B$, $\forall x.A$, $\exists x.A$, if A, B formulas, $x \in var_{analysis} \cup VAR_{analysis}$.
A Δ^1_0 formula is a formula, not containing bounded second-order quantifiers, and a Π^1_1-formula is $\forall X.\phi$, where ϕ is a Δ^1_0-formula.

Assumption 7.2 *After renaming all variables, we assume, we have additional new variables U_i of KPi^+ ($i \in \omega$) and Z_i of ML ($i \in \omega$), s.t. in the step from a g-type to A^*, Z_i becomes U_i and in 4.7 (a), if $x_i = Z_j$, then on the right side we put U_j.*

Definition 7.3 *(a) Let $\lceil C_i^{set} \rceil$ be new Gödelnumbers for new construstructors C_i^{set}.*

(b) Let for a set b, $Embset_{ML}(f) := \{< n, \underline{n}_0 > | n \in \omega \setminus b\} \cup \{< n, \underline{n}_1 > | n \in \omega \cap b\}$.

(c) If $CES(b)$, the constructors in b are $\neq \lceil C_i^{set} \rceil$, $b_1, \ldots, b_m \in Ad_2$, $b_i \subset \omega$, $Z_{i_j} \neq Z_{i_k}$, ($j \neq k$), then
$a_0(b, Z_{i_1}/b_1, \ldots, Z_{i_m}/b_m) := b \cup \{< \lceil C_{i_1}^{set} \rceil, 0, f_1 >, \ldots, < \lceil C_{i_m}^{set} \rceil, 0, f_m >\}$,
where $f_i := Embset_{ML}(b_i)$.

Definition 7.4 *Let $\mathcal{P}(N) := N \to U$, $ML_1^i W_T \vdash N \to U$ type.*

Definition 7.5 *(a) For each primitive recursive $g : I\!N^k \to I\!N$ we define a closed g-term $int_{PA,ML}(g)$, (we abbreviate this as $\hat{g} := int_{PA,ML}(g)$) such that*

$$ML \vdash \hat{g} : \underbrace{N \to \cdots N}_{k \ times} \to N,$$

and we define a set $int_{PA,KP}(g)$ short $\widetilde{g}$ in L_{KP}
such that $KPi^+ \vdash fun(\widetilde{g}) \wedge dom(\widetilde{g}) = I\!N^k \wedge \forall x \in I\!N^k . \widetilde{g}(x) \in I\!N$.
Case $g = S$: $\hat{g} := \lambda x.Sx$, $\widetilde{g} := \{< x, x+1 > | x \in I\!N\}$.
Case $g = Proj_i^n$ $\quad \hat{g} := \lambda x_1, \ldots, x_n.x_i$, $\widetilde{g} := \{<< x_1, \ldots, x_n >, x_i > | x_1, \ldots, x_n \in I\!N\}$.
Case $g = Cons_c^n$:
$\hat{g} := \lambda x_1, \ldots, x_n.S^c 0$, $\widetilde{g} := \{<< x_1, \ldots, x_n >, c > | x_1, \ldots, x_n \in I\!N\}$.
Case $g(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$:
$\hat{g} := \lambda x_1, \ldots, x_n.\hat{h}(\hat{g}_1 x_1 \cdots x_n) \cdots (\hat{g}_m x_1 \cdots x_n)$,
$\widetilde{g} := \{<< x_1, \ldots, x_n >, h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)) >$
$| x_1, \ldots, x_n \in I\!N\}$.
Case $g(x_1, \ldots, x_n, 0) = h(x_1, \ldots, x_n)$,
$g(x_1, \ldots, x_n, y+1) = k(x_1, \ldots, x_n, y, g(x_1, \ldots, x_n, y))$:
$\hat{g} := \lambda x_1, \ldots, x_n, y.P(y, \hat{h} x_1 \cdots x_n, (u, v)\hat{k} x_1 \cdots x_n uv)$,
define $l(x_1, \ldots, x_n, 0) := \widetilde{h}(x_1, \ldots, x_n)$,
$l(x_1, \ldots, x_n, Sy) := \widetilde{k}(x_1, \ldots, x_n, y, l(x_1 \cdots x_n, y))$,
then
$\widetilde{g} := \{<< x_1, \ldots, x_n, y >, l(x_1, \ldots, x_n, y) > | x_1, \ldots, x_n, y \in I\!N\}$.

(b) For each term t of analysis we define a g-term $int_{PA,ML}(t)$, short $\hat{t}$ and a term of L_{KP} $int_{PA,KP}(t)$, short $\widetilde{t}$, such that, if $FV(t) = \{v_{i_1}, \ldots, v_{i_n}\}$ ($i_1 < \cdots < i_n$) (v_i as in definition 7.1 of $var_{analysis}$) then $FV(\hat{t}) \subset \{z_{i_1}, \ldots, z_{i_n}\}$, $FV(\widetilde{t}) \subset \{u_{i_1}, \ldots, u_{i_n}\}$ and $ML \vdash z_{i_1} : N, \ldots, z_{i_n} : N \Rightarrow \hat{t} : N$, and
$KPi^+ \vdash \forall u_{i_1}, \ldots, u_{i_n} \in \omega.(\widetilde{t} \in \omega)$.
Case $t = v_i$: $\hat{t} := z_i$, $\widetilde{t} := u_i$.
Case $t = 0$: $\hat{t} := 0$, $\widetilde{t} := 0$.
Case $t = gt_1 \cdots t_n$: $\hat{t} := \hat{g}\hat{t}_1 \cdots \hat{t}_n$, $\widetilde{t} := \widetilde{g}(\widetilde{t}_1, \ldots, \widetilde{t}_n)$.

(c) For each formula A of analysis we define a g-type $int_{PA,ML}(A)$ (short $\widehat{A}$), and a formula of L_{KP} $int_{PA,KP}(A)$, short $\widetilde{A}$, such that in KPi^+ $\widetilde{A}$ is equivalent to a Δ_0-formula, and if $FV(A) = \{v_{i_1}, \ldots, v_{i_n}, V_{j_1}, \ldots, V_{j_m}\}$, $i_k \neq i_l$, $j_k \neq j_l$ for $k \neq l$, then $FV(\widehat{A}) \subset \{z_{i_1}, \ldots, z_{i_n}, Z_{j_1}, \ldots, Z_{j_m}\}$, $FV(\widetilde{A}) \subset \{u_{i_1}, \ldots, u_{i_n}, U_{j_1}, \ldots, U_{j_m}\}$ and for

all four versions of Martin-Löf Type Theory ($ML, ML_1^iW_T, ML_1^eW_R, ML_1^iW_R$
$ML \vdash z_{i_1} : N, \ldots, z_{i_n} : N, Z_{j_1} : \mathcal{P}(N), \ldots, Z_{j_m} : \mathcal{P}(N) \Rightarrow \widehat{A}$ *type.*
Case $A = (s = t)$: $\widehat{A} := I(N, \hat{s}, \hat{t})$, $\widetilde{A} := (\widetilde{s} = \widetilde{t})$.
Case $A = (t \in V_i)$: $\widehat{A} := T(Z_i\hat{t})$ *for the Tarski-version* $\widehat{A} := Z_i\hat{t}$ *for the Russell-version,* $\widetilde{A} := \widetilde{t} \in U_i$.
Case $A = (B \wedge C)$: $\widehat{A} := (\widehat{B} \times \widehat{C})$, $\widetilde{A} := \widetilde{B} \wedge \widetilde{C}$.
Case $A = (B \vee C)$: $\widehat{A} := (\widehat{B} + \widehat{C})$, $\widetilde{A} := (\widetilde{B} \vee \widetilde{C})$.
Case $A = (B \to C)$: $\widehat{A} := (\widehat{B} \to \widehat{C})$, $\widetilde{A} := (\widetilde{B} \to \widetilde{C})$.
Case $A = \forall v_i.B$: $\widehat{A} := \Pi z_i \in N.\widehat{B}$, $\widetilde{A} := \forall u_i \in \omega.\widetilde{B}$.
Case $A = \exists v_i.B$: $\widehat{A} := \Sigma z_i \in N.\widehat{B}$, $\widetilde{A} := \exists u_i \in \omega.\widetilde{B}$.
Case $A = \forall V_i.B$: $\widehat{A} := \Pi Z_i \in \mathcal{P}(N).\widehat{B}$, $\widetilde{A} := \forall U_i \in Ad_2.U_i \subset \omega \to \widetilde{B}$.
Case $A = \exists V_i.B$: $\widehat{A} := \Sigma Z_i \in \mathcal{P}(N).\widehat{B}$, $\widetilde{A} := \exists U_i \in Ad_2.U_i \subset \omega \wedge \widetilde{B}$.
Case $A = \perp$: $\widehat{A} := N_0$, $\widetilde{A} := (0 \neq 0)$.

Definition 7.6 *(a) We define* $emb : \omega \to \omega$, $embnat(n) := S^n0 (=: \hat{n})$ *(or more precisely* $\lceil S^n 0 \rceil$*), a function definable in* KPi^+.

(b) $< a, \cdot > := < a, a >$.

Lemma 7.7 *(a) If* $g : I\!N^k \to I\!N$ *is primitive recursive, then*

$$KPi^+ \vdash \quad \forall t_1, \ldots, t_k \in Term_{Cl}.\forall n_1, \ldots, n_k. \\ (r^1 \to_{red} \hat{\hat{n}}_1 \wedge \cdots \wedge r^k \to_{red} \hat{\hat{n}}_k) \to \hat{g}r_1, \ldots, r_k \to_{red} emb(\widetilde{g}(n_1, \ldots, n_k)).$$

(b) If t is a term of analysis, $FV(t) \subset \{v_1, \ldots, v_n\}$*, then*

$$KPi^+ \vdash \quad \forall r_1, \ldots, r_n \in Term_{Cl}.\forall n_1, \ldots, n_k.(r_1 \to_{red} \hat{\hat{n}}_1 \wedge \cdots \wedge r_k \to_{red} \hat{\hat{n}}_k) \to \\ \hat{t}[z_1/r_1, \ldots, z_n/r_n] \to_{red} emb(\widetilde{t}[u_1/n_1, \ldots, u_n/n_n]).$$

Proof: (a) Case $g = S$: $\hat{g}r_1 \to_{red} (\lambda x.Sx)\hat{\hat{n}}_1 \to_{red} S\hat{\hat{n}}_1 = emb(Sn_1)$.
Case $g = Proj_i^n$: $\hat{g}t_1, \ldots, r_n \to_{red} \hat{\hat{n}}_i = emb(\widetilde{g}(n_1, \ldots, n_k))$.
Case $g = Cons_c^n$: trivial.
Case $g(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x}))$, $l_i := \widetilde{g}_i(n_1, \ldots, n_n)$:

$$\hat{g}_i(r_1, \ldots, r_n) \to_{red} emb(\widetilde{g}_i(n_1, \ldots, n_n)) =: emb(l_i),$$

therefore

$$\begin{aligned} & \hat{g}r_1, \ldots, r_n \\ = \quad & \hat{h}\hat{g}_1(r_1, \ldots, r_n), \ldots, \hat{g}_m(r_1, \ldots, r_n) \\ \to_{red} \quad & \hat{h}\hat{l}_1, \ldots, \hat{l}_n \\ \to_{red} \quad & emb(\widetilde{h}(l_1, \ldots, l_m)) \\ = \quad & embnat(g(n_1, \ldots, n_n)) \end{aligned}$$

Case $g(x_1, \ldots, x_n, 0) = h(x_1, \ldots, x_n)$, $g(\vec{x}, x_{n+1}) = k(\vec{x}, y, g(\vec{x}, x_{n+1})))$. Let $\hat{h}(r_1, \ldots, r_n) \to_{red} embnat(\widetilde{h}(n_1, \ldots, n_n)) =: l_0$. We show by induction on n_{n+1}, that $P(\hat{\hat{n}}_{n+1}, l_0, (\lambda u, v.\hat{k}r_1, \ldots, r_nuv)) \to_{red} emb(\widetilde{g}n_1, \ldots, n_{n+1})$
(since $g(r_1, \ldots, r_n) \to_{red} P(\hat{\hat{n}}_{n+1}, l_0, (\lambda u, v.\hat{k}r_1, \ldots, r_nuv))$ follows the assertion)
If $n_{n+1} = 0$,

$$\begin{aligned} & P(nhathat_{n+1}, l_0, (\lambda u, v.\hat{k}r_1, \ldots, r_nuv)) \\ \to_{red} \quad & l_0 \\ = \quad & emb(\widetilde{g}(n_1, \ldots, n_n, n_{n+1})) \end{aligned}$$

If $n_{n+1} = m+1$,

$$\begin{array}{rl} & g(t_1,\ldots,t_n,t_{n+1}) \\ \rightarrow_{red} & P(S\hat{\hat{m}}, l_0, (\lambda u, v.\hat{k}r_1,\ldots,r_n uv)) \\ \rightarrow_{red} & (\lambda u, v.\hat{k}r_1,\ldots,r_n uv))\hat{\hat{m}}P(\hat{k}, l_0, (\lambda u, v.\hat{k}r_1,\ldots,r_n uv)) \\ \rightarrow_{red} & (\lambda u, v.\hat{k}r_1,\ldots,r_n uv))\hat{\hat{m}}emb(g(n_1,\ldots,n_n,m) \\ \rightarrow_{red} & \hat{k}r_1,\ldots,r_n\hat{\hat{m}}emb(g(n_1,\ldots,n_n,m) \\ \rightarrow_{red} & \hat{k}\hat{\hat{n}}_1,\ldots,\hat{\hat{n}}_n\hat{\hat{m}}emb(g(n_1,\ldots,n_n,m) \\ \rightarrow_{red} & emb(\widetilde{k}(n_1,\ldots,n_n mg(n_1,\ldots,n_n,m)) \\ = & emb(\widetilde{g}(n_1,\ldots,n_{n+1}) \end{array}$$

(b): If $t = v_i, 0$, this is trivial,
and if $t = gt_1,\ldots,t_n$ follows

$$t_i[\vec{r}]\rightarrow_{red} emb(\widetilde{t}_i[\vec{n}]),$$

by (a) therefore

$$t[\vec{r}]\rightarrow_{red}\widetilde{g}(\widetilde{t}_1[\vec{r}],\ldots,\widetilde{t}_n[\vec{r}]) = \widetilde{t}[\vec{r}]$$

Next task would now be to prove, that, when we first interpret a formula of $L_{analysis}$ in L_{ML} and then use the interpretation, as we have done in section 4, we get an equivalent formula to the one, we get by directly interpreting $L_{analysis}$ in L_{KP}. But in this formulation, this is not correct, here is the place, where we need to extend the set of term constructors by non constructive constructors. In order to interpret a true Π^0_2 formula $A = \forall x.\exists y.\phi$ in such a way, that for the false formula $\neg A$ we have $(\neg A)^* = \emptyset$ we need an element of A^*, which gives for the x a witness y of ϕ. But this might be non constructive, so we add here a (possibly) non constructive new constructor. So for every formula we need certain new constructors. Further, we want that Π^1_1-formulas are interpreted correctly as well, that is, we want, that if we have a free set variable V_i, we can replace it in the KPi^+-interpretation as arbitrary subsets $U_i \subset \omega$, $U_i \in Ad_2$. We achieve this by allowing here arbitrary interpretations for the constructor C^{set}_i.

Lemma 7.8 *For every Δ^1_0-formula A with*

$$FV(A) \subset \{v_{i_1},\ldots,v_{i_l},V_{j_1},\ldots,V_{j_m}\}$$

with $i_k \neq i_l$, $j_k \neq j_l$, $(k \neq l)$ there is a CES c not referring to the constructors C^{set}_i $(i \in \omega)$ and a g-term $h \in Term_{Cl}$ s.t. $FV(h) \subset \{z_{i_1},\ldots,z_{i_l}\}$, and with $\vec{z} := z_{i_1},\ldots,z_{i_l}$, $\vec{Z} := Z_{j_1},\ldots,Z_{j_m}$, $\vec{u} := u_{i_1},\ldots,u_{i_l}$, $\vec{U} := U_{j_1},\ldots,U_{j_m}$, $\vec{C^Z} := C^{set}_{j_1},\ldots,C^{set}_{j_m}$, $\vec{n} := n_1,\ldots,n_l$, $Emb\vec{s}et(b) := Embset(b_1,\ldots,b_m)$, we have

$$\begin{array}{ll} KPi^+_n \vdash & \forall n_1,\ldots,n_l \in \omega.\forall r_1,\ldots,r_l \in Term_{Cl}. \\ & \forall b_1,\ldots,b_m \in Ad_2.\forall a_0 \in Ad_2. \\ & b_1 \subset \omega \rightarrow b_m \subset \omega \rightarrow CES(a_0) \wedge \\ & a_0(c,b_1,\ldots,b_m) \subset a_0 \rightarrow \\ & (r_1\rightarrow_{red}\hat{\hat{n}}_1 \rightarrow \cdots \rightarrow r_l\rightarrow_{red}\hat{\hat{n}}_l) \rightarrow \\ & \quad ((\exists r \in Term_{nf}.h[\vec{z}/\vec{r}]\rightarrow_{red} r) \wedge \\ & \quad (\widetilde{A}[\vec{u}/\vec{n},\vec{U}/\vec{]}\leftrightarrow < h[\vec{z}/\vec{r}],\cdot > \in \widehat{A}^*[\vec{z}/\vec{r}]) \wedge \\ & \quad (\widetilde{A}[\vec{u}/\vec{n}] \leftrightarrow \widehat{A}^*[\vec{z}/\vec{r}] \neq \emptyset)) \end{array}$$

Proof: by induction on the definition of the formulas.
Again we will not mention explicitly Variables, that occur in subterms, or do not occur at all.

Case $A = \perp$: Choose $c := \emptyset$, $h := 0$ We have $\neg\widetilde{A}[\vec{n}]$, $\widehat{A}^*[\vec{r}] = \emptyset$.

Case $A = (s = t)$: Choose as $c := \emptyset$, $h := \underline{r} \in Term_{nf}$. We have, using that for $r \longrightarrow_{red} s \in Term_{nf}$ s is unique, and 7.7

$$\begin{aligned}
\widetilde{A}[\vec{n}] &\leftrightarrow \widetilde{s}[\vec{n}] = \widetilde{t}[\vec{n}] \\
&\leftrightarrow (\exists n \in \omega . \hat{s}[\vec{r}] \longrightarrow_{red} S^n 0 \wedge \hat{t}[\vec{r}] \longrightarrow_{red} S^n 0) \\
&\leftrightarrow < \hat{s}[\vec{r}], \hat{t}[\vec{r}] > \in N^* \\
&\leftrightarrow < \underline{r}, \underline{r} > \in \widehat{A}^*[\vec{r}] \\
&\leftrightarrow \widehat{A}^*[\vec{r}] \neq \emptyset
\end{aligned}$$

Case $A = (s \in V_i)$: Let $c := \emptyset$, $h := 0_1$. Assume $\vec{r}$, $\vec{n}$, b_i, a_0 as in the assumption. Then $\hat{s}[\vec{z}/\vec{r}] \longrightarrow_{red} emb(\widetilde{s}[\vec{u}/\vec{n}]) = S^k 0$ for some k.
$\widehat{A}[\vec{z}/\vec{r}, \vec{Z}/\vec{C^Z}]^*_{a_0} = T(C_i^{set}(\overline{s}[\vec{z}/\vec{r}]))_{a_0}$
$= \begin{cases} T(\underline{n}_0) & \text{if } k \notin b_i \\ T(\underline{n}_1) & \text{if } k \in b_i \end{cases} = \begin{cases} \emptyset & \text{if } k \notin b_i \\ Compl(\{0_1\}) & \text{if } k \in b_i \end{cases}.$
$\widetilde{A}[\vec{u}/\vec{n}, \vec{U}/\overrightarrow{Embset}(b)] = \widetilde{s}[\vec{u}/\vec{n}] \in Embset(b_i)$. Then the assertion is immediate.

Case $A = (A_1 \wedge A_2)$: Let c_i, h_i for A_i chosen, $c := c_1 \cup c_2$, $h := p(h_1, h_2)$. Then for $\vec{n}, \vec{r}, \vec{b}, a_0$ as in the assumption of the assertion there exist $s_1, s_2 \in Term_{nf}$, such that $r_i[\vec{r}] \longrightarrow_{red} s_i$, $h[\vec{r}] \longrightarrow_{red} p(s_1, s_2) \in Term_{nf}$.

$$\begin{aligned}
\widetilde{A}[\vec{n}] &\leftrightarrow \widetilde{B}_1[\vec{n}] \wedge \widetilde{B}_2[\vec{n}] \\
&\leftrightarrow < s_1, \cdot > \in \widehat{B}_1^*[\vec{n}] \wedge < s_2, \cdot > \in \widehat{B}_2^*[\vec{n}] \\
&\leftrightarrow < h[\vec{r}], \cdot > \in \widehat{A}^*[\vec{n}] \\
&\leftrightarrow \widehat{B}_1^*[\vec{r}] \neq \emptyset \wedge \widehat{B}_2^*[\vec{r}] \neq \emptyset \\
&\leftrightarrow \widehat{A}^*[\vec{n}] \neq \emptyset
\end{aligned}$$

Case $A = (B_1 \vee B_2)$: Let c_i, h_i for B_i chosen. Let

$$f := \{<< n_1, \ldots, n_l >, i > |(i = 0 \wedge \widetilde{B}_0[\vec{n}]) \vee (i = 1 \wedge \neg\widetilde{B}_0[\vec{n}])\}$$

(note that $\widetilde{B}_0$ is a Δ_0-formula).
Let $\lceil C \rceil$ be a Gödel-number for a new constructor, different from all $\lceil C^{set} \rceil$, $c := c_1 \cup c_2 \cup \{< \lceil C \rceil, l+1, f\}$, $h := P(C\vec{z}, h_1, (u, v)h_2)$ (u, v new variables). Assume $\vec{n}$, $\vec{r}$, $\vec{b}$, a_0 as in the assumption. Then there exist s_i such that $h_i[\vec{r}] \longrightarrow_{red} s_i \in Term_{nf}$

$$C(r_1, \ldots, r_n) \longrightarrow_{red} C(\hat{\hat{n}}_1, \ldots, \hat{\hat{n}}_i) \longrightarrow_{red} S^i 0$$

for $i = f(\vec{n}) \in \{0, 1\}$, if $i = 0$,

$$h[\vec{r}] \longrightarrow_{red} P(0, s_1, (u, v)h_2[\vec{r}]) \longrightarrow_{red} s_1 \in Term_{nf},$$

and if $i = 1$,

$$h[\vec{r}] \longrightarrow_{red} s_2.$$

We have

$$\begin{aligned}
\widetilde{A}[\vec{n}] &\leftrightarrow \widetilde{B}_1[\vec{n}] \vee (\neg\widetilde{B}_1[\vec{n}] \wedge \widetilde{B}_2[\vec{n}]) \\
&\leftrightarrow (f(\vec{n}) = 0 \wedge < s_1, \cdot > \in \widehat{B}_1^*[\vec{r}]) \vee (f(\vec{n}) = 1 \wedge < s_2, \cdot > \in \widehat{B}_2^*[\vec{r}]) \\
&\leftrightarrow < h[\vec{r}], \cdot > \in \widehat{A}^*[\vec{r}] \\
&\leftrightarrow \widehat{B}_1^*[\vec{r}] \neq \emptyset \vee \widehat{B}_2^*[\vec{r}] \neq \emptyset \\
&\leftrightarrow \widehat{A}^*[\vec{r}] \neq \emptyset
\end{aligned}$$

Case $A = (B_1 \to B_2)$. Let c_i, h_i for B_i chosen, $c := c_1 \cup c_2$, $h := \lambda x.h_2$. Assume $\vec{r}$, $\vec{n}$, $\vec{b}$, a_0 as in the assumption. Then $h_2[\vec{r}] \to_{red} s_2$ for some $s_2 \in Term_{nf}$.
Subcase $\widetilde{A}[\vec{n}]$. If $\widetilde{B}_1[\vec{n}]$ is false, then by IH $\widehat{B}_1^*[\vec{r}] = \emptyset$, therefore

$$\forall < r, r' > \in \widehat{B}_1^*[\vec{r}]. < h_2[x/r, \vec{r}], h_2[x/r', \vec{r}] > \in \widehat{B}_2^*[\vec{r}]$$

therefore $< h[\vec{r}], \cdot > \in \widehat{A}^*[\vec{r}]$.
If $\widetilde{B}_1[\vec{n}]$ is true, then $\widetilde{B}_2[\vec{r}]$ is true, therefore $< s_2, \cdot > \in \widehat{B}_2^*[\vec{r}]$,

$$\forall < r, r', \cdot > \in \widehat{B}_1^*[\vec{r}]. h_2[x/r, \vec{r}] \to_{red} s_2 \wedge h_2[x/r', \vec{r}] \to_{red} s_2 \wedge < s_2, \cdot > \in \widehat{B}_2^*[\vec{n}],$$

$h[\vec{r}] \in \widehat{A}^*[\vec{n}]$.
Subcase $\neg\widetilde{A}[\vec{n}]$. Then by IH exists s_1 such that $h_1[\vec{r}] \to_{red} s_1 \in Term_{nf}$ and we have $< s_1, \cdot > in \widehat{B}_1^*[\vec{r}]$ and, if we had $< s, s' > \in \widehat{A}^*[\vec{r}]$, then $< s, \cdot > \in \widehat{A}^*[\vec{r}]$, $s \to_{red} \lambda x.t$ for some t, $< t[x/s_1], \cdot > \in \widehat{B}_2^*[\vec{r}] = \emptyset$, a contradiction, therefore $\widehat{A}^*[\vec{r}] = \emptyset$.

Case $A = \forall v_i.B$: Let c_1, h_1 for B be chosen, $c := c_1$, $h := \lambda v_i.h_1$. Assume $\vec{n}$, $\vec{r}$, $\vec{b}$, a_0 as in the assertion, $h[\vec{r}] \in Term_{nf}$.
Assume $< r, r' > \in \widehat{A}^*[\vec{r}]$, then $< r, r > \in \widehat{A}^*[\vec{r}]$, $r \to_{red} \lambda x.t$ and

$$\forall k \in \omega < r[x/\hat{k}, \vec{r}], \cdot > . \in \widehat{B}^*[z_i/\hat{k}, \vec{r}],$$

by IH follows $\forall k \in \omega.\widetilde{B}[u_i/k, \vec{n}]$, therefore $\widetilde{A}[\vec{n}]$.
Assume $\widetilde{A}[\vec{n}]$. Then for all $k \in \omega$ $\widetilde{B}[v_i/k, \vec{r}]$, therefore by I.H. $< h[v_i/r, \vec{r}], \cdot > \in \widehat{B}^*[z_i/r, \vec{r}]$, whenever $r \to_{red} S^k 0$, therefore $< h[\vec{r}], \cdot > \in \widehat{A}[\vec{r}]$.

Case $A = \exists v_i.B$, c_1, h_1 be chosen for B,

$$\begin{aligned}
f := \{<< \vec{n} >, k > \mid \quad & (\widetilde{B}[u_i/k, \vec{n}] \wedge \forall k' < k.\neg(\widetilde{B}[u_i/k, \vec{n}])) \vee \\
& (k = 0 \wedge \forall k \in \omega.\neg(\widetilde{B}[u_i/k, \vec{n}]))\}
\end{aligned}$$

Let $\lceil C \rceil$ be a new name for a constructor $\neq C_i^{set}$, $c := c_1 \cup \{< \lceil C \rceil, l+1, f\}$. $h := p(C(\vec{z}), h_1[z_i/C(\vec{z})])$ Assume $\vec{n}$, $\vec{r}$ $\vec{b}$, a_0 as in the assertion, $k := f(\vec{n})$.

$$C(\vec{r}) \to_{red} C(\hat{n}_1, \ldots, \hat{n}_n) \to_{red} S^k 0.$$

By IH we have $h_1[z_i/C(\vec{z})][\vec{r}] = h_1[z_i/C(\vec{r}), \vec{r}] \to_{red} t_1$ for some $t_1 \in Term_{nf}$, therefore $h[\vec{r}] \to_{red} p(S^k 0, t_1)$.
Assume $< r, r' > \in \widehat{A}^*[\vec{r}]$. Then $< r, r > \in \widehat{A}^*[\vec{r}]$, $r \to_{red} p(S^l 0, r'') \in Term_{nf}$. Then $< r'', \cdot > \in \widehat{B}^*[z_i/S^l 0, \vec{r}]$, by I.H. $\widetilde{B}[u_i/l, \vec{n}]$, therefore $\widetilde{A}[\vec{n}]$.
Assume $\widetilde{A}[\vec{n}]$. Then by definition $\widetilde{B}[u_i/k, \vec{n}]$ and by IH

$$< t_1, \cdot > \in \widehat{B}^*[z_i/\hat{k}, \vec{r}] = \widehat{B}^*[z_i/A_k(\vec{r}), \vec{r}] = \widehat{B}[z_i/A_k(\vec{z})]^*[\vec{z}/\vec{r}],$$

(we use that $ML \vdash \vec{z} : N \Rightarrow A\ type$,
$< A_k(\vec{r}), \hat{k} > \in N^*$ and the Main Lemma), therefore $< h[\vec{r}], \cdot > \in \widehat{A}^*[\vec{r}]$.

Lemma 7.9 *If ϕ is a Π^1_1-formula, $ML \vdash s : \widehat{\phi}$, then $KPi^+ \vdash \widetilde{\phi}$.*
Proof: Let $\phi = \forall V_i.B$. $ML \vdash s : \widehat{\phi}$. By 6.1 follows $KPi^+ \vdash \forall CES(b). < \hat{s}^*_b, \cdot > \in \widehat{\phi}^*_b$. $\widehat{\phi}^* = \Pi^*(\widehat{N \to U^*}, \widehat{B})$. Assume $b_1 \in Ad_2$, x a variable, c the CES chose for B as in lemma 7.8, $a_0 := b_0 \cup \{< C^{set}_i, 0, Embset(b) >\}$. $C_i \in (N \to U)^*_{a_0}$. Therefore, under the assumption $b_i \in Ad_2$, $b_i \subset \omega$, $KPi^+ \vdash < \hat{s}^* C^{set}_i, \cdot > \in \widehat{B}^*_{a_0}[V_i/C^{set}_i]$, by lemma 7.8 $KPi^+ \vdash \widetilde{B}[U_i/b_i]$, and we have $KPi^+ \vdash \forall V_i \in Ad_2.V_i \subset \omega \to \widetilde{B}$ which is $\widetilde{\phi}$. **We are now coming to the end** and just have to prove, that the result we have suffices to show, that we have really an upper bound for the proof-theoretical strength. We have to overcome the fact, that we did only prove, that if $ML \vdash TI(\prec)$ ($TI(\prec)$ for transfinite induction over a primitive recursive relation $\prec$), we get $KPi^+ \vdash TI^{Ad_2}(\prec)$, where TI^{Ad_2} means transfinite induction, with the quantifier over subsets of ω, which are elements of Ad_2. But we will see, that this will be enough:
Definition 7.10 *We define some formulas in L_{KP}:*

(a) In the following, $(a, \prec)$ will be a pair where a is a set, and $\prec \subset a \times a$. In this context $s \prec t := < s, t > \in \prec$, $\forall x \prec t.\phi := \forall x \in a.x \prec t \to \phi$, and $\exists x \prec t.\phi := \exists x \in a.x \prec t \wedge \phi$. Further $s \preceq t := s \prec t \vee s = t$.

(b) $Wf^d(a, \prec) := \prec \subset a \times a \wedge \forall x \in d.x \subset a \to x \neq \emptyset \to \exists y \in x.\forall z \prec y.z \notin x$. ($\prec$ is a relation on a which is well-founded, restricted to d).

(c) $Collaps(a, \prec, f) := Fun(f) \wedge dom(f) = a \wedge \forall x \in a.f(x) = \{f(y)|y \prec x\}$. ($f$ is a collapsing function on $(a, \prec)$.

Lemma 7.11 *If $\phi(y, y_1, \ldots, y_n)$ is a Δ_0-formula with only the free variables mentioned, then*
$KPi^+ \vdash Wf^d(a, \prec) \to Ad(c) \to \forall y_1, \ldots, y_n \in c.(\forall x \in a.(\forall y \prec x.\phi(y, y_1, \ldots, y_n) \to \phi(x, y_1, \ldots, y_n))) \to \forall x \in a.\phi(x, y_1, \ldots, y_n)$. The formula after the second arrow is called principle of restricted induction over $(a, \prec)$.
Proof: Immediate.
Lemma 7.12 *$KPi^+ \vdash Ad(c) \to Ad(d) \to c \in d \to Wf^d(a, \prec) \to c, d \in a \to \exists f \in c.Collaps(a, \prec, f)$.*
Proof:
As in [Jäg86], theorem 4.6, but replacing Δ_0-induction by d-induction.
Lemma 7.13 *If $KPi^+ \vdash \forall x.Ad(x) \to \phi(x)$ for a Σ_1-formula ϕ^x, then $L_v \models \phi$, where ϕ^x is the restriction of all unrestricted quantifiers to x, and $v := \psi_{\Omega_1}(\Omega_{I+\omega})$.*
Proof:
We follow the lines of [Buc92]. First observe, that we can prove as in theorem 2.9 there, using several applications of $\exists_\kappa$, $(\bigwedge)$, and $\vdash^* Ad(L_\kappa)$ for $\kappa \in R$, and if we have $\lambda \in Lim$, $(\kappa_i)_{i \in \omega}$ a sequence, s.t. $\kappa_0 \in R$, $\forall \alpha \in \kappa.\exists \rho \in \kappa.\alpha \in \rho \in R$ and $\kappa_i \in \kappa_{i+1} \in R \cap \lambda$ ($i \in \omega$), and if we extend X^* by including as well κ_i ($i \in \omega$) follows

$$\vdash^*_\lambda (KPi^+)^\lambda.$$

($\vdash^*_\lambda \phi^\lambda$ for every axiom ϕ of KPi^+). We can adjust theorem 3.12 of [Buc92] to obtain, if we have in this situation, if $\lambda \in \mathcal{H}$, $\kappa_i \in \mathcal{H}$ ($i \in \omega$) and $\mathcal{H}$ closed under $\xi \mapsto \xi^R$, then:

For each theorem ϕ of KPi^+ exists $k \in \mathbb{N}$ such that $\mathcal{H} \vdash^{\omega^{\lambda+k}}_{\lambda+k} \phi^\lambda$.

Now observe, that $\mathcal{H}_\gamma$ in [Buc92] has the desired properties (with $\lambda := \Omega_{I+\omega}$, $\kappa_i := \Omega_{I+i}$) and we conclude that if $KPi^+ \vdash \forall x.Ad(x) \to \phi^x$, where ϕ is a Σ-sentence, then $L_v \models \phi$ for $v := \psi_{\Omega_1}(\Omega_{I+\omega})$.

Theorem 7.14 *$|ML|$, $|ML|$, $|ML_1^eW_{R,U}|$, $|ML_1^eW_R|$, $|ML_1^iW_T|$, $|ML_1^iW_R| \leq v$ with $v := \psi_{\Omega_1}(\Omega_{I+\omega})$, where the ordinal denotation is as in [Buc92].*

Proof: Since all the theories can be embedded in ML in such a way, that the principle of transfinite induction remains unchanged (except, that $T(U_it)$ becomes U_it in the version à là Russell), it is sufficient, to prove it for ML. Assume $\prec$ is a primitive recursive linear ordering on the primitive recursive subset T of ω, $\phi := \forall X.(\forall y.y \in T \rightarrow (\forall z.z \prec y \rightarrow z \in X) \rightarrow y \in X) \rightarrow \forall y.y \in T \rightarrow y \in X$ and $ML \vdash \widehat{\phi}$. Then by lemma 7.9 $KPi^+ \vdash \widetilde{\phi}$. We follow the proof of [Rat91] theorem 7.14, define $a := \{x \in \omega | x \in T\}$, $prec' := \{< x, y >\in \omega \times \omega | x \prec y\}$. Then $KPi^+ \vdash Ad(Ad_1) \wedge Ad(Ad_2) \wedge Ad_1 \in Ad_2 \wedge Wf^{Ad_2}(a, \prec') \wedge a, \prec' \in Ad_1$, therefore by 7.12 $KPi^+ \vdash \exists f \in Ad_1.Collaps(a, \prec', f)$, $KPi^+ \vdash \forall x.Ad(x) \rightarrow \exists f \in x.Collaps(a, \prec', f)$. Therefore $L_v \models \exists f.Collaps(a, \prec', f)$. Since $\prec$ is linear ordering, follows $Image(f)$ is an ordinal, and, because $v \in Lim$ we have $Image(f) \in L_v$, $ordertype(\prec) = Image(f) < v$.

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